# Kriging Demystified

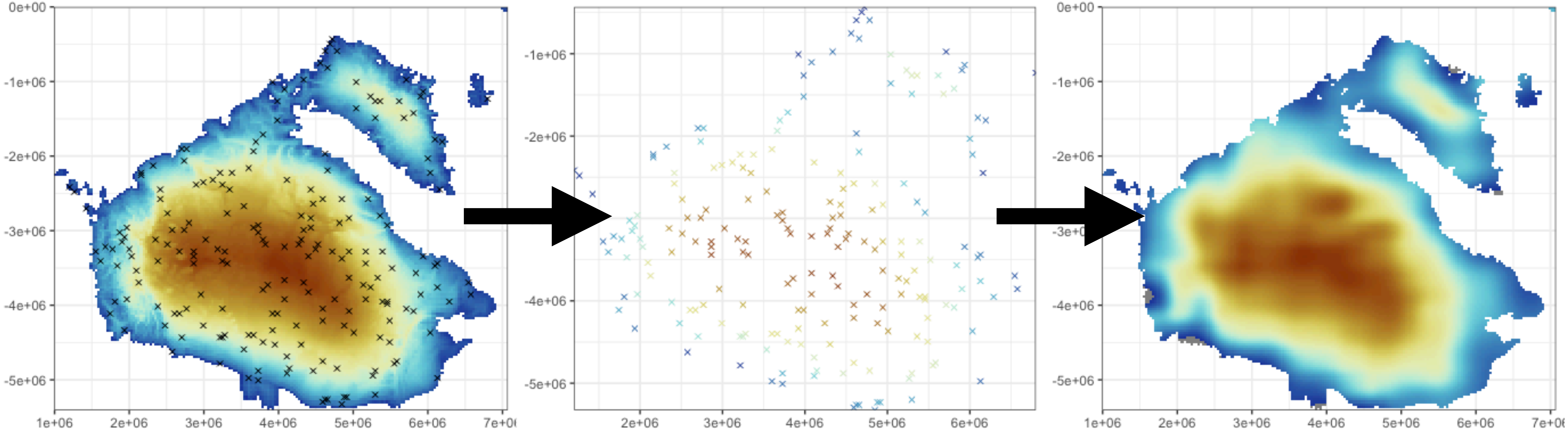
## What’s going on under the hood of our favourite interpolator?

**Lachlan Astfalck**

# What is it, why do we care?

Kriging is one of the most influential interpolation methods in statistics, geosciences, engineering, and anything else that needs interpolation.

If you’ve asked SPSS or GIS to interpolate something before it’s probably using Kriging or something related.



The multiple regression procedure for arriving at the ***best linear unbiased*** [predictor] or ***best linear weighted moving average*** [predictor] of the ore grade of an ore block (of any size) by assigning an optimum set of weights to all the available and relevant data inside and outside the ore block

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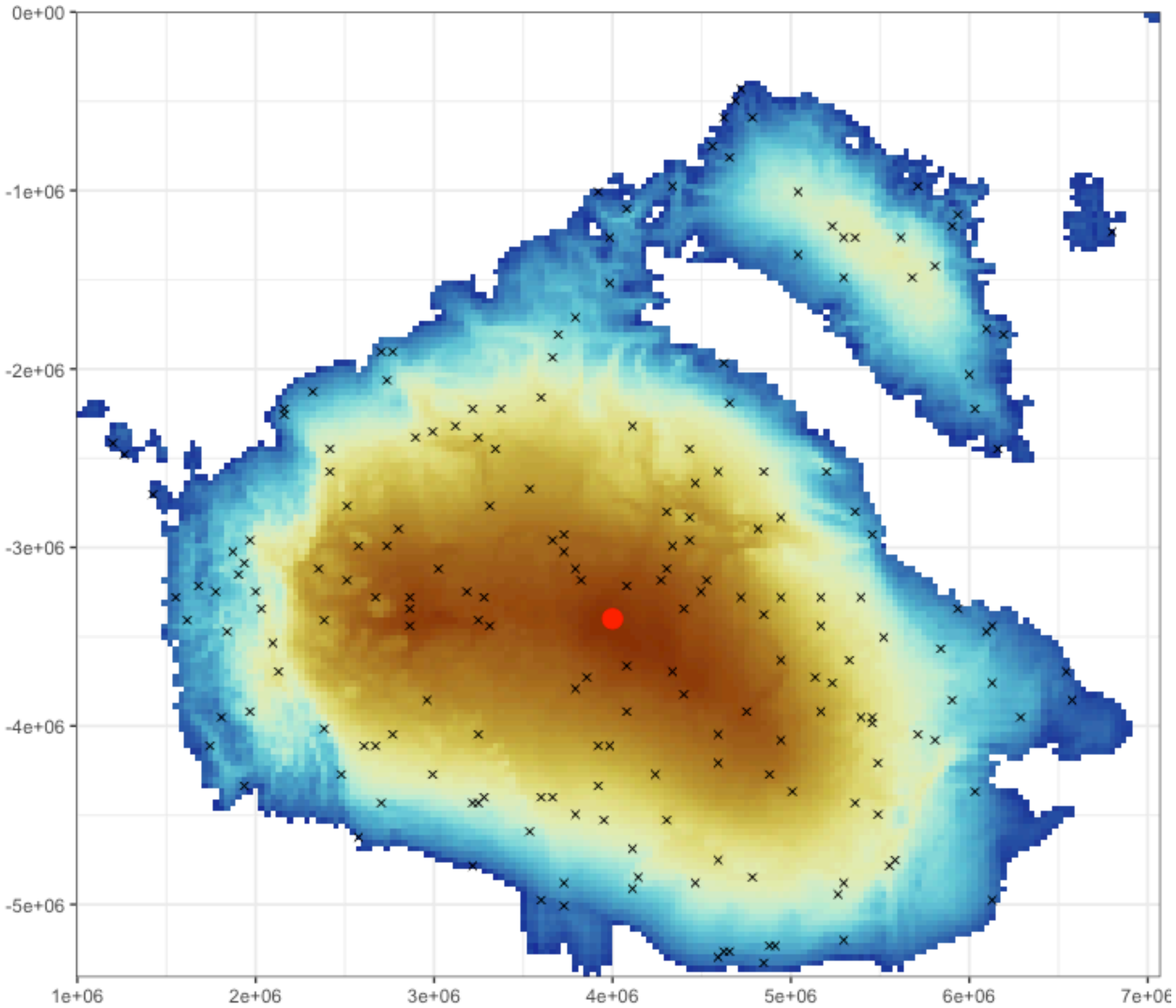
***“Everything is related to everything else, but near things are more related than distant things” -*** Tobler’s First Law of Geography

# Building up interpolation



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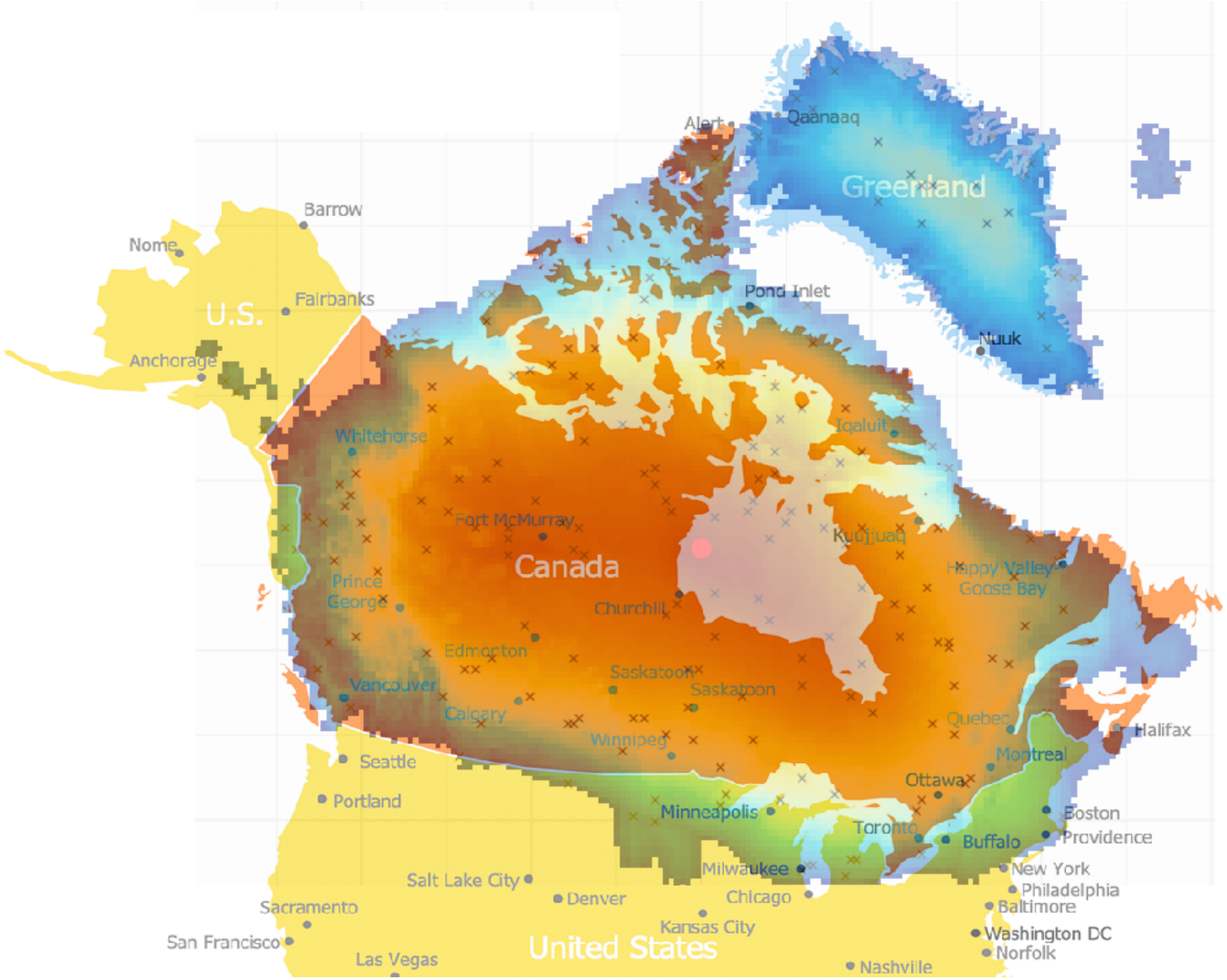

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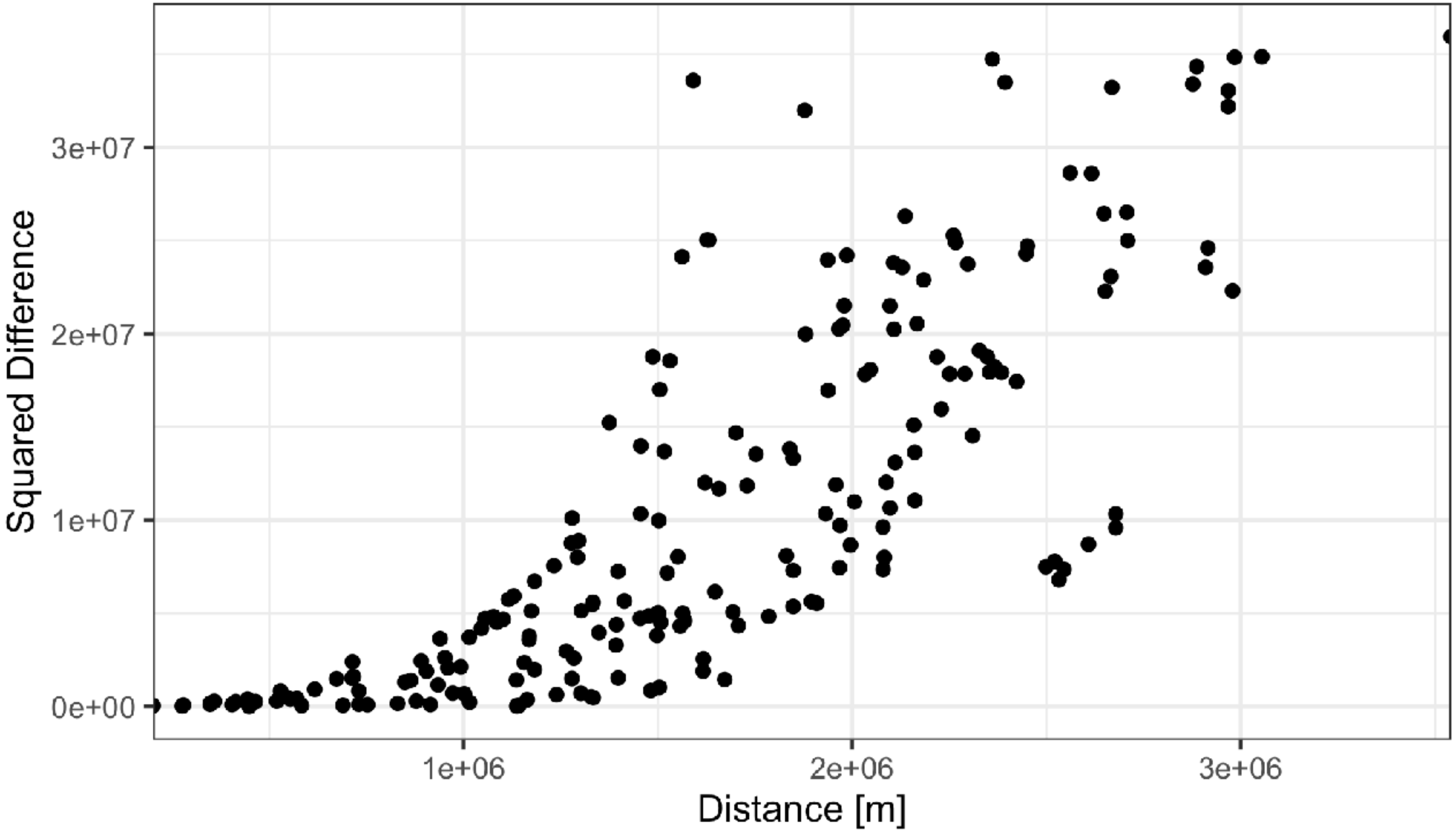
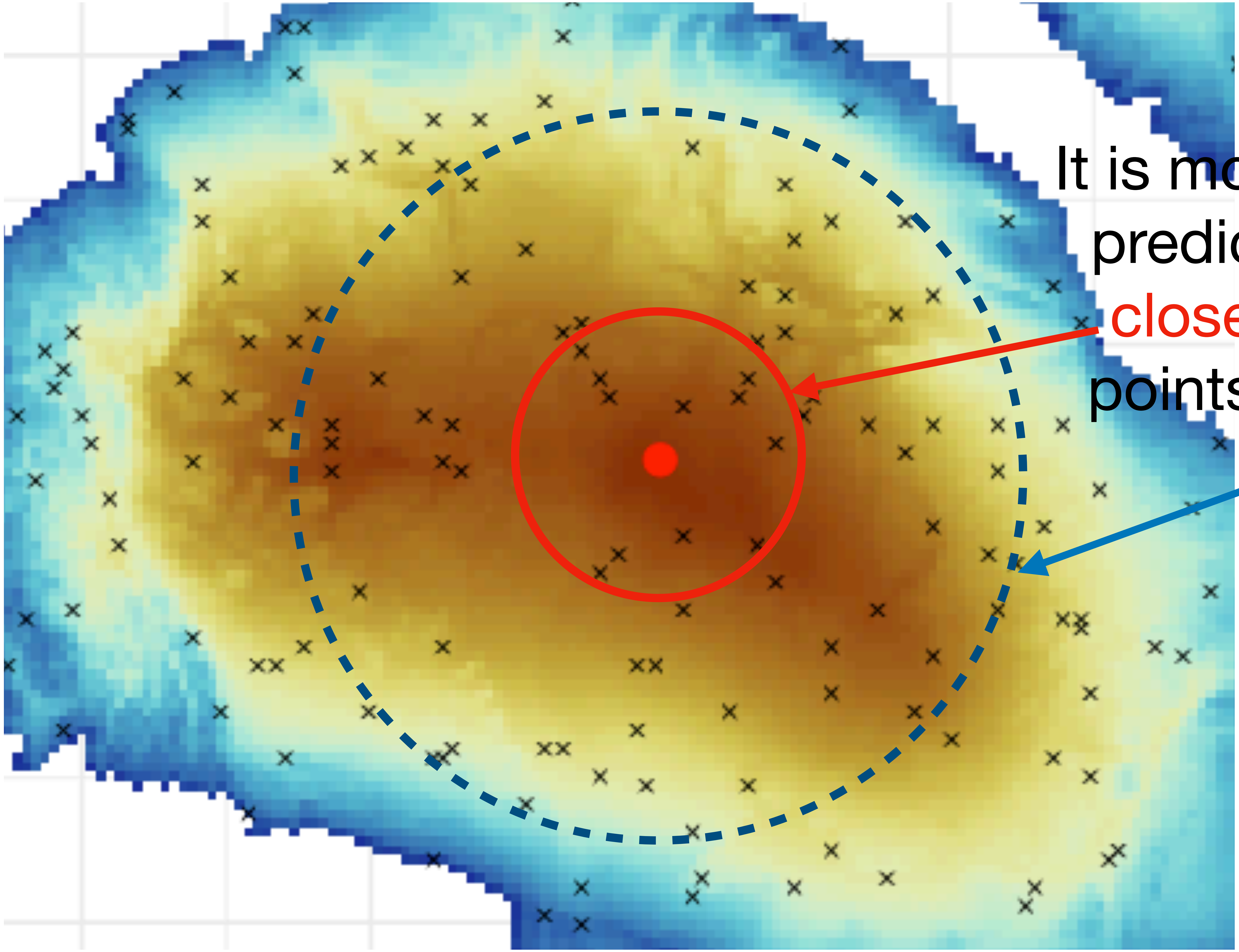



Tobler's First Law


It is more desirable to predict from points closer rather than points further away

# Sensible interpolation methods include…

Regression methods

Bi-linear interpolation

Inverse distance weighting

Nearest neighbour predictions

Spline interpolation

**Kriging**

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We're going to assume all data have point support.

# Cranking out the math

We want the ***best linear unbiased predictor*** of $Z(s_0)$. This is given by a weighted linear average of the observed data that minimises the mean-squared error

$$\mathbb{E}\left(\left(Z(s_0) - \sum_{i=1}^{n} a_i Z(s_i) - k\right)^2\right)$$

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Minimising this objective yields

$$\mathrm{a} = \mathrm{c}' C^{-1} \qquad k = E\left(Z(s_0)\right) + \mathrm{c}' C^{-1} E\left(\mathrm{Z}\right)$$

where $\mathrm{c} \equiv (C(s_0, s_1), \ldots, C(s_0, s_n))'$ and $C_{i,j} \equiv C(s_i, s_j)$.

(we're about to go through why…)

# Cranking out the math

We may thus say that the optimal predictor of $Z(s_0)$ is

$$Z^*(s_0) = E\left(Z(s_0)\right) + \mathrm{c}'C(\mathrm{Z} - E\,(\mathrm{Z}))$$

with prediction error

$$E\left(\left(Z(s_0) - Z^*(s_0)\right)^2\right) = C(s_0, s_0) - \mathrm{c}'C^{-1}\mathrm{c}$$

# Cranking out the math

First we need some basic identities:

$$E\left(x^2\right) = \text{var}(x) + \left(E(x)\right)^2$$

$$\text{var}(x + y) = \text{var}(x) + 2\text{cov}(x, y) + \text{var}(y)$$

$$\text{cov}(ax, y) = a\text{cov}(x, y)$$

$$\text{var}(ax) = a^2\text{var}(x)$$

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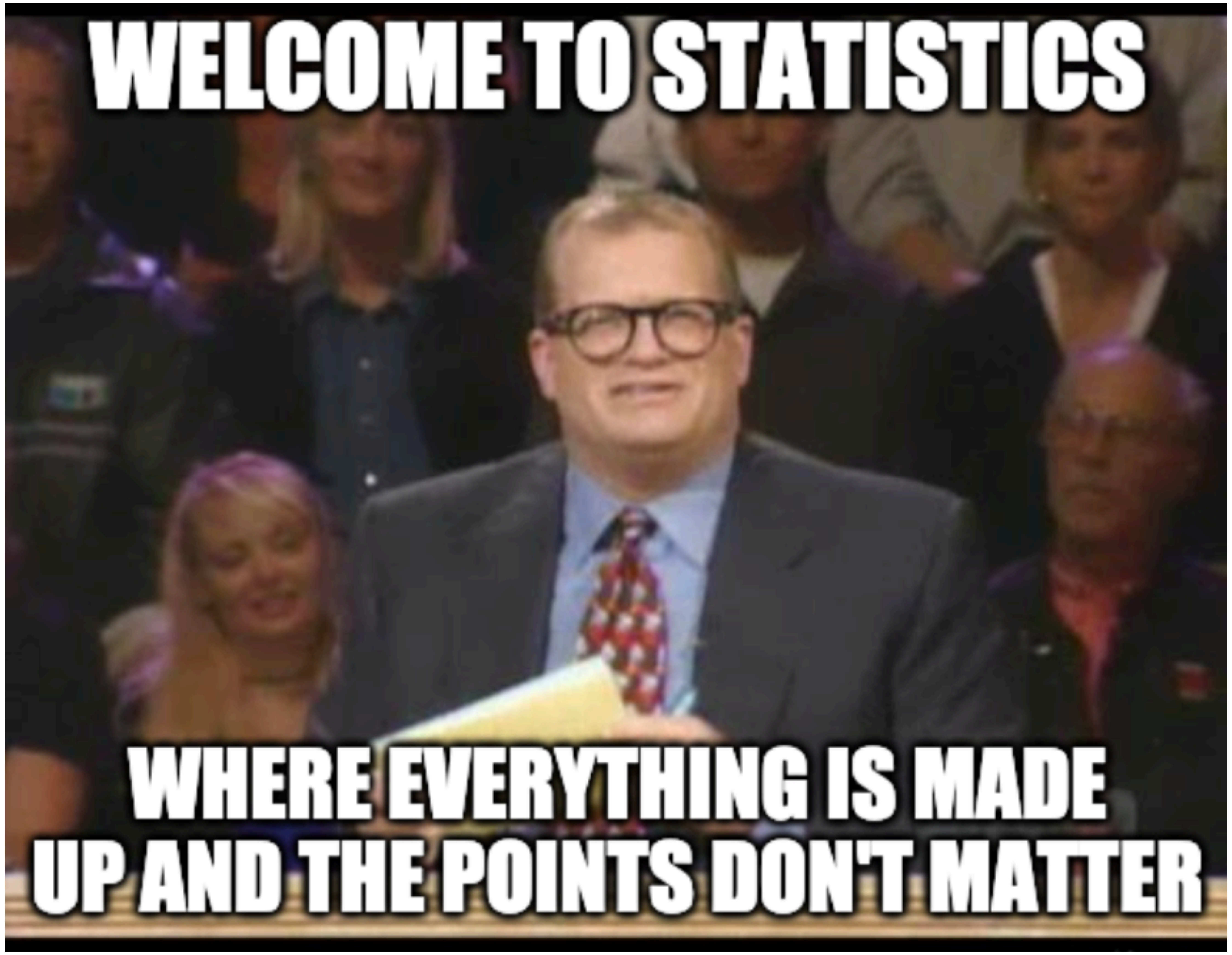
and so $\mathrm{a}^T = \mathrm{cov}\left(Z(s_0), \mathrm{Z}\right)\mathrm{var}\left(\mathrm{Z}\right)^{-1}$

# But how do we figure out covariance?



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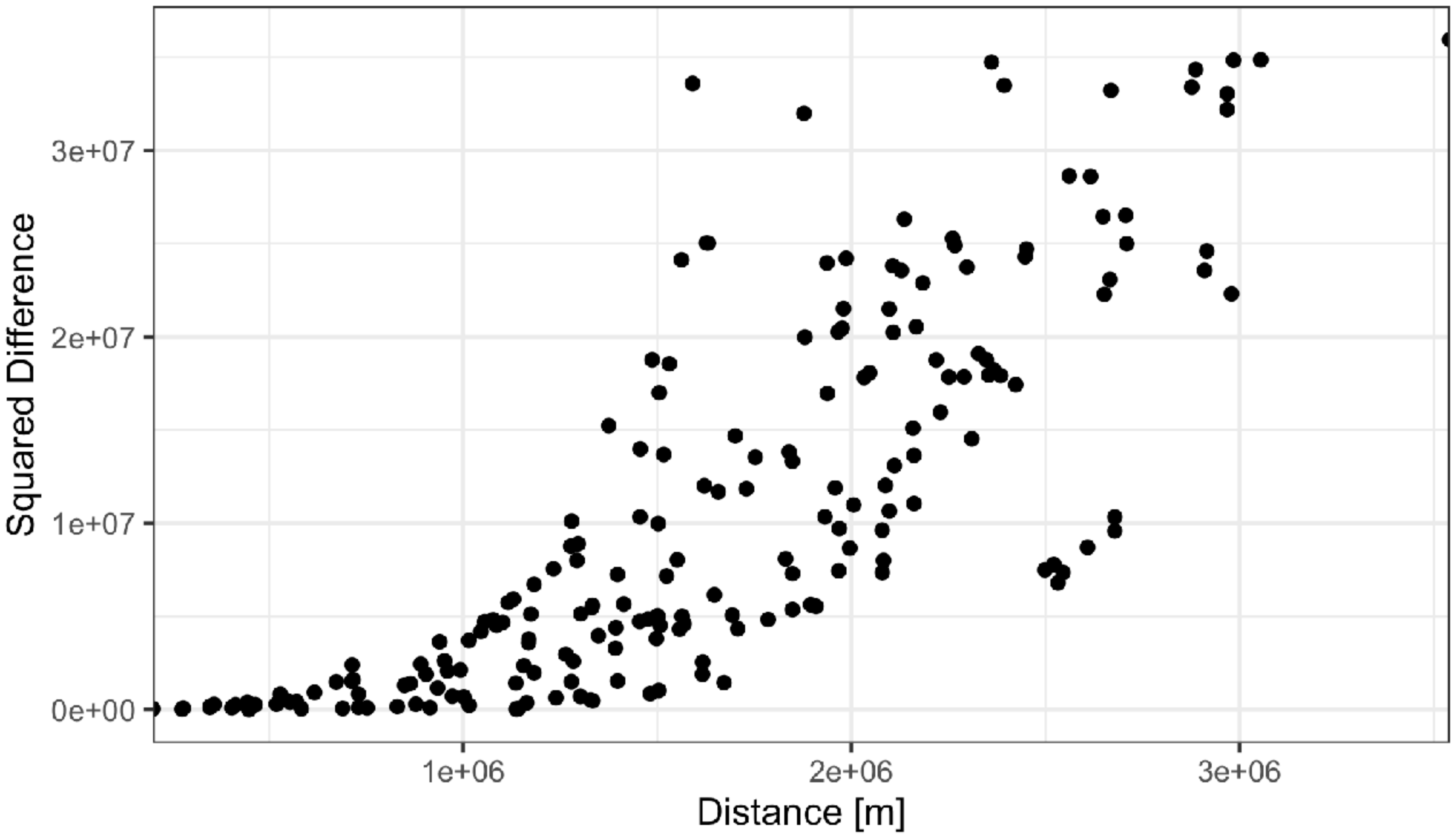

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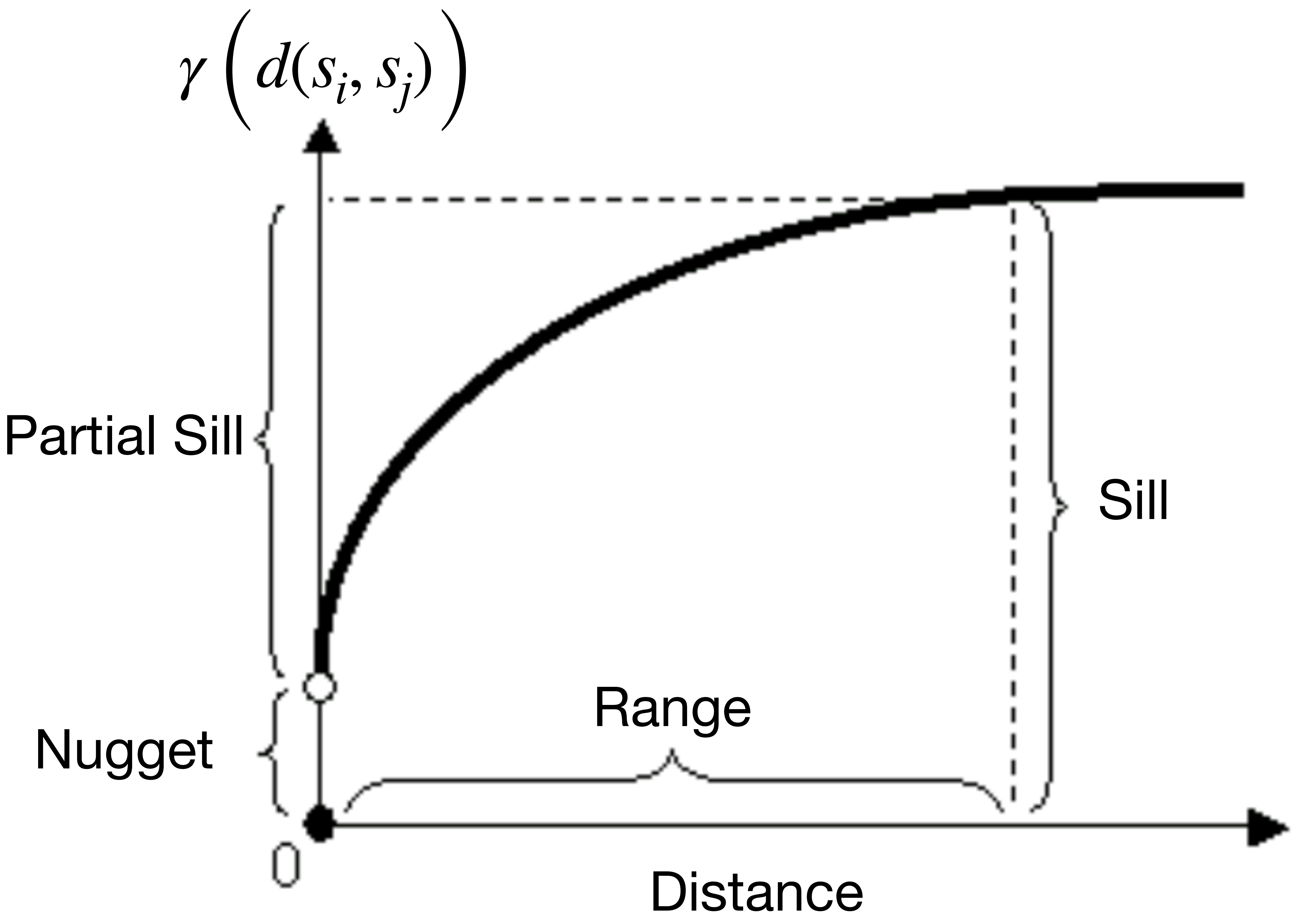
The covariance function describes how two points co-vary as a function of distance

# The semivariogram



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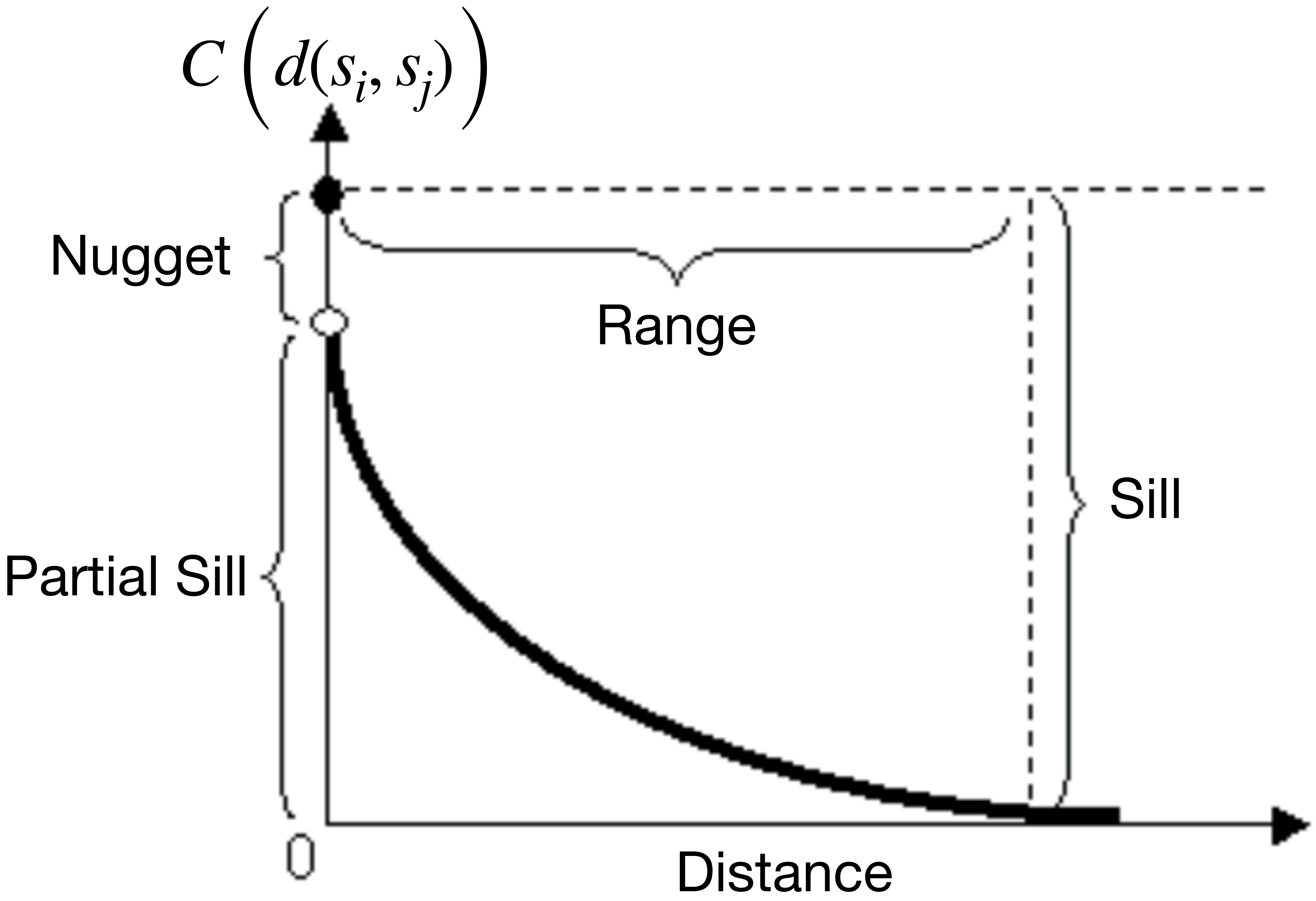
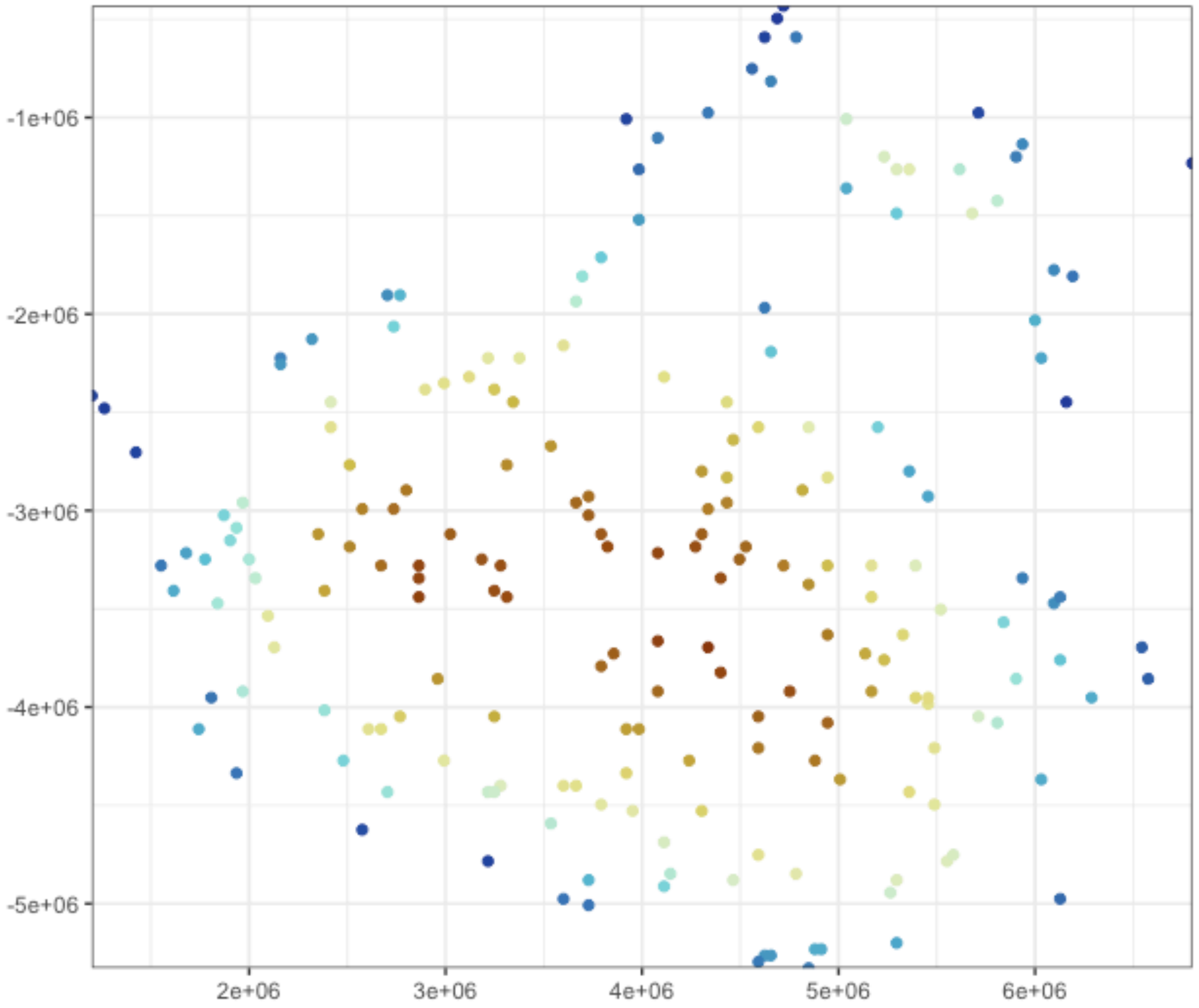


# Empirical Semivariogram




Once we have a fit semivariogram or covariance function we're good to go

# Empirical Semivariogram

$$\hat{\gamma}(h) = \frac{1}{n_h} \sum_{s_i, s_k \in N(h)} \left( Z\left(s_i\right) - Z\left(s_k\right) \right)^2$$

for some choice of $h = | s_i - s_k |$

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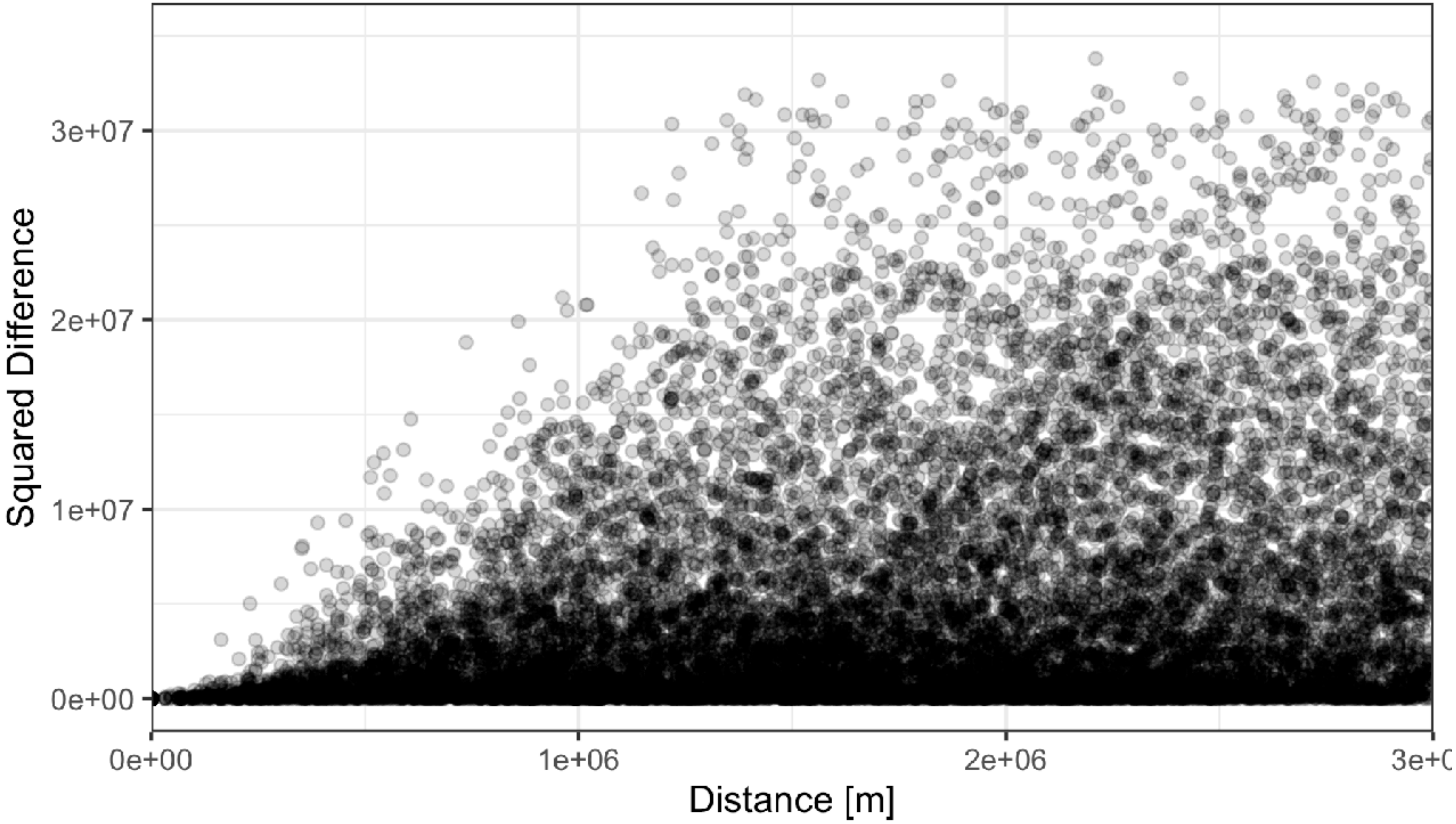

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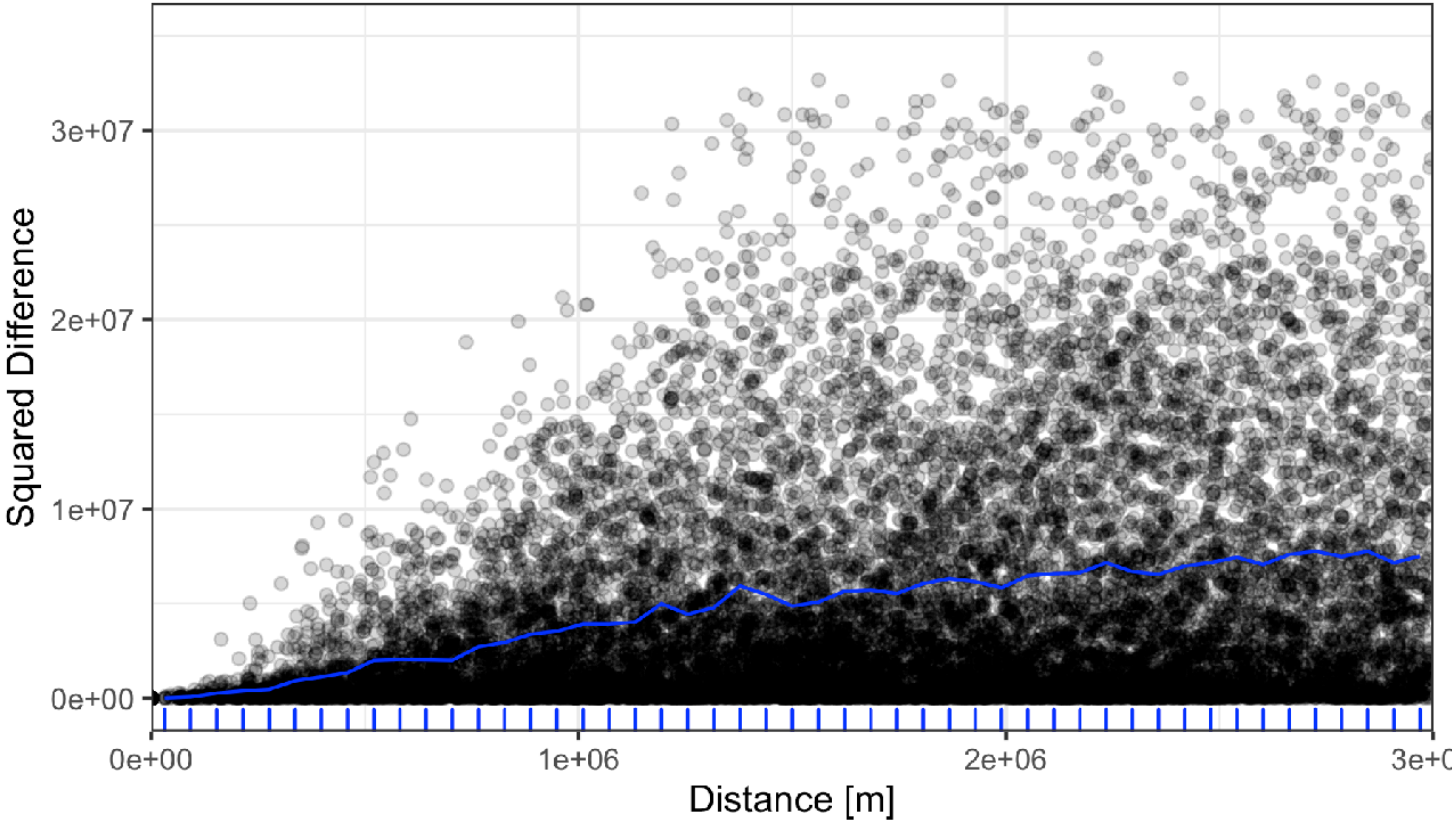

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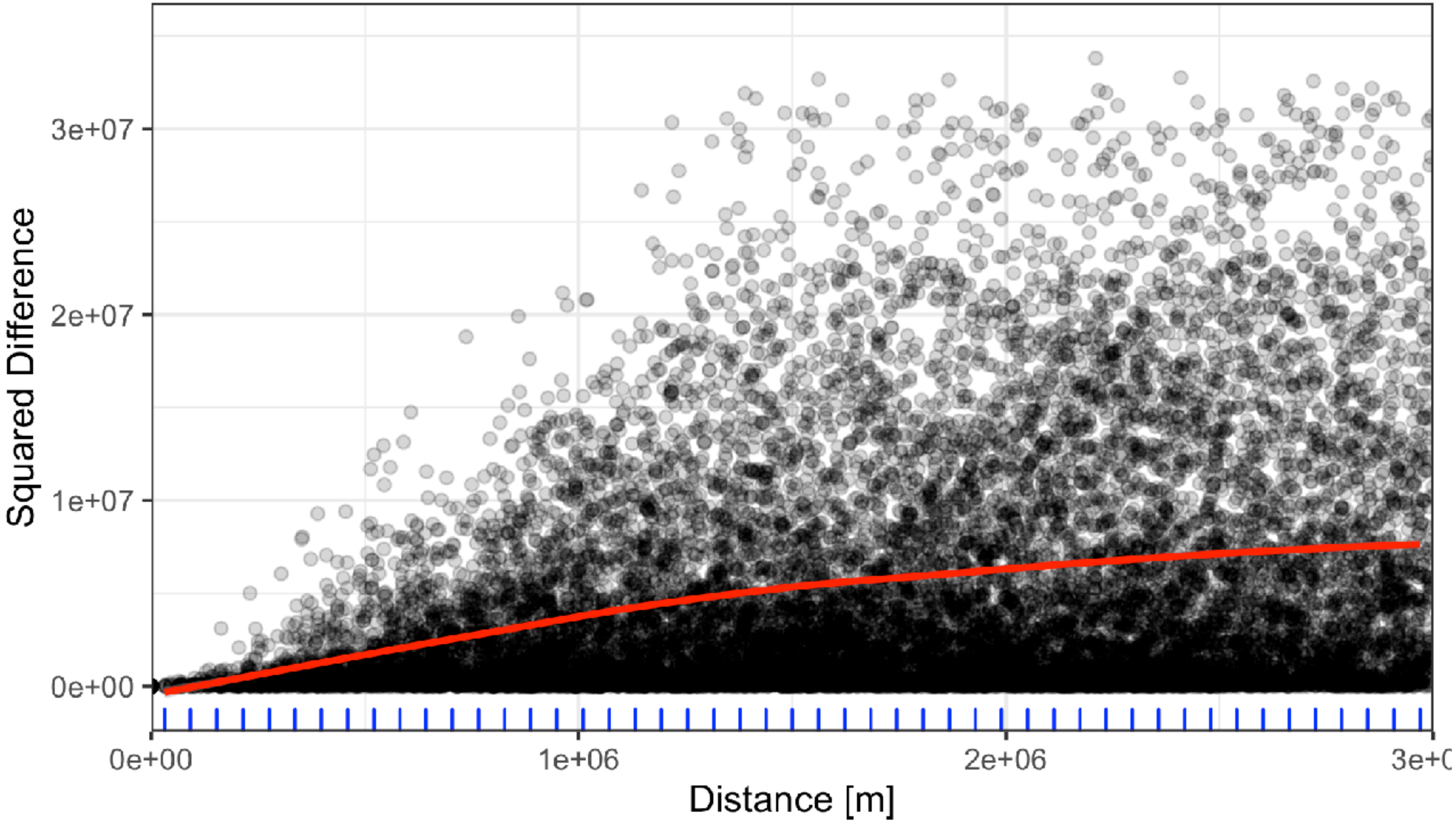

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Be aware of what covariance function is being assumed - different functions yield drastically different interpolations.

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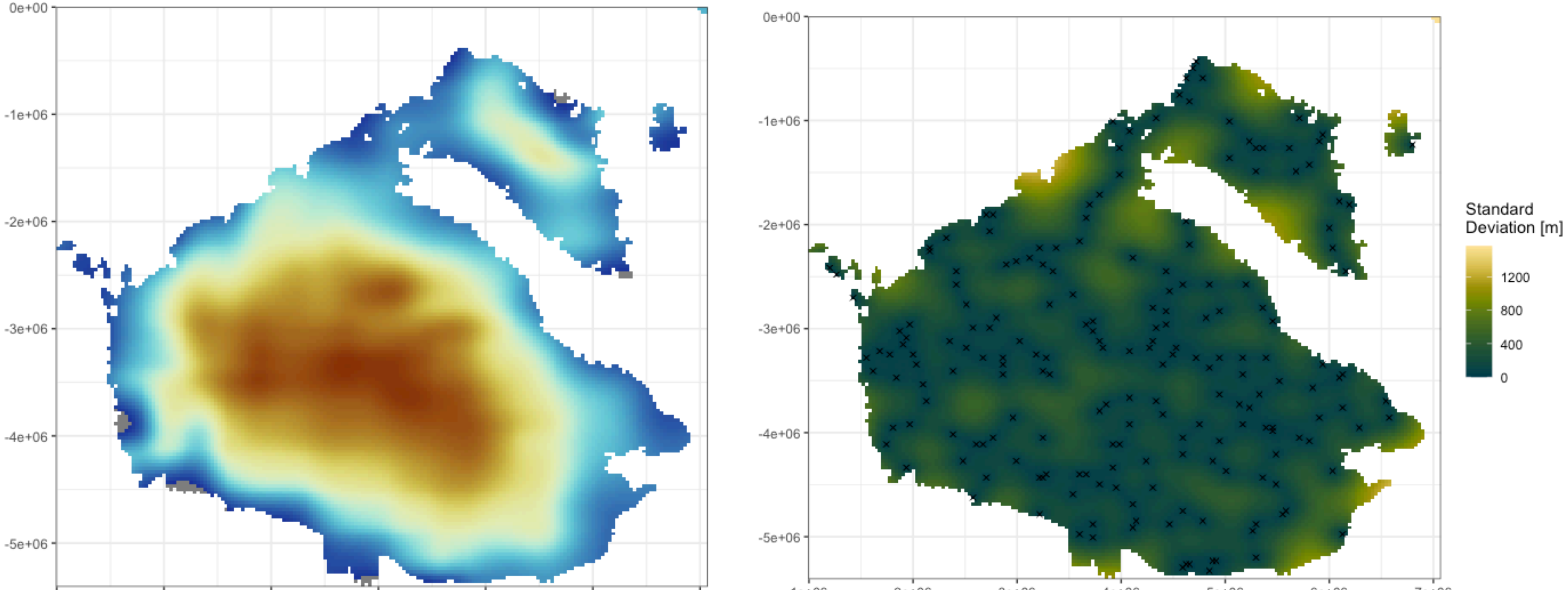


$E\left(Z^*(s)\right)$ $\mathrm{var}\left(Z^*(s)\right)$

# A rose by any other name…

Simple Kriging

Ordinary Kriging

Universal Kriging

Disjunctive Kriging

Linear Regression

Gaussian Processes

Kalman Filtering

Gaussian Markov Random Fields

Wiener-Kolmogorov Predictions

Spline Regression in Reproducing Kernel Hilbert Spaces

Bayes linear analysis

# A rose by any other name…

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Bayes linear analysis

The essence/ideas underpinning kriging are everywhere! If you’re ever interpolating or regressing, be mindful of where it may be.

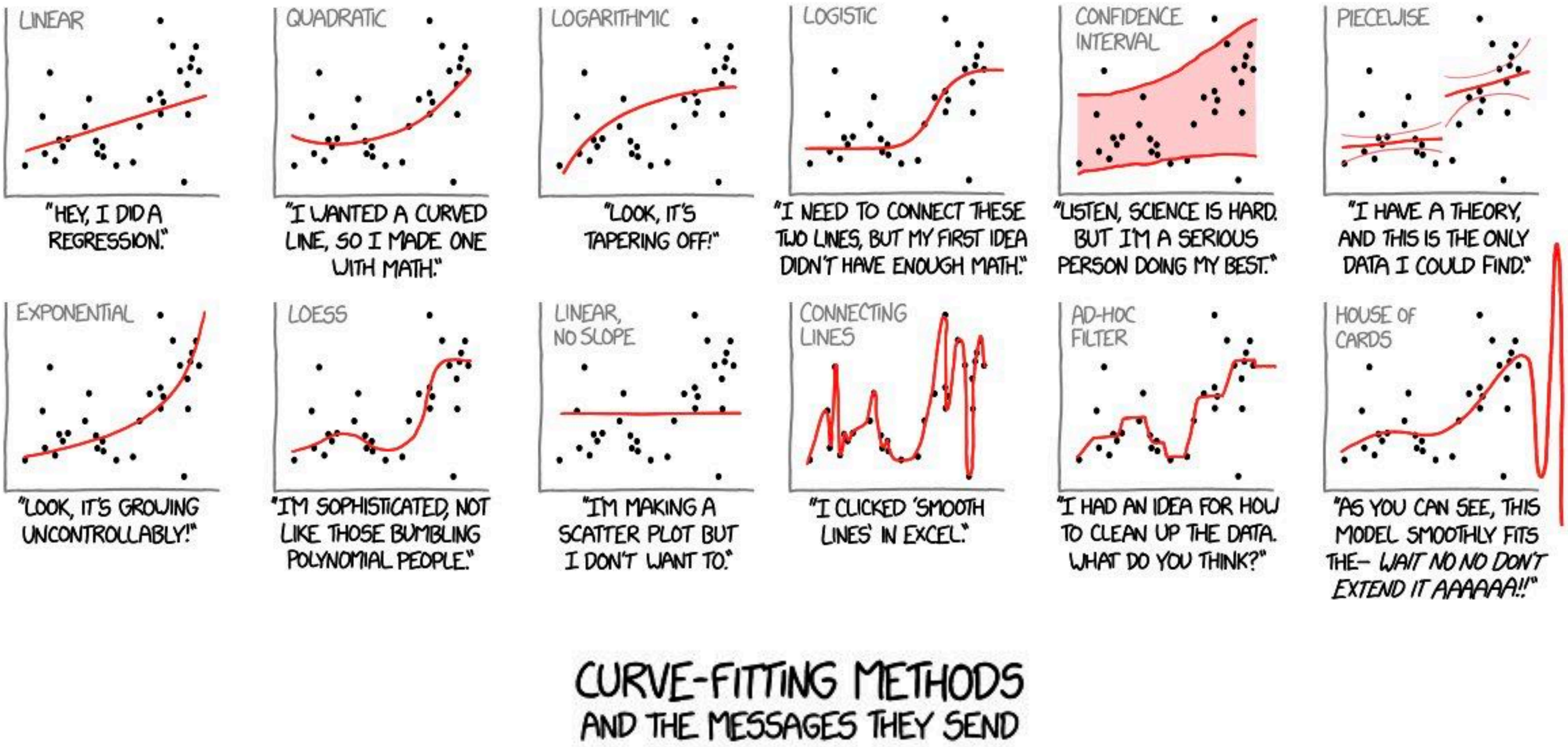
# What we haven’t talked about

Stationarity, types of covariance functions and positive definiteness

Methods of inference

Kriging with big data

Assumptions of Gaussinity


LINEAR
"HEY, I DID A REGRESSION."
QUADRATIC
"I WANTED A CURVED LINE, SO I MADE ONE WITH MATH."
LOGARITHMIC
"LOOK, IT'S TAPERING OFF!"
LOGISTIC
"I NEED TO CONNECT THESE TWO LINES, BUT MY FIRST IDEA DIDN'T HAVE ENOUGH MATH."
CONFIDENCE INTERVAL
"LISTEN, SCIENCE IS HARD. BUT I'M A SERIOUS PERSON DOING MY BEST."
PIECEWISE
"I HAVE A THEORY, AND THIS IS THE ONLY DATA I COULD FIND."
EXPONENTIAL
"LOOK, IT'S GROWING UNCONTROLLABLY!"
LOESS
"I'M SOPHISTICATED, NOT LIKE THOSE BUMBLING POLYNOMIAL PEOPLE."
LINEAR, NO SLOPE
"I'M MAKING A SCATTER PLOT BUT I DON'T WANT TO."
CONNECTING LINES
"I CLICKED 'SMOOTH LINES' IN EXCEL."
AD-HOC FILTER
"I HAD AN IDEA FOR HOW TO CLEAN UP THE DATA. WHAT DO YOU THINK?"
HOUSE OF CARDS
"AS YOU CAN SEE, THIS MODEL SMOOTHLY FITS THE— WAIT NO NO DON'T EXTEND IT AAAAAA!!"
CURVE-FITTING METHODS
AND THE MESSAGES THEY SEND