Reconstructing Boundary Conditions for Complex Simulations with Application to Climate Models

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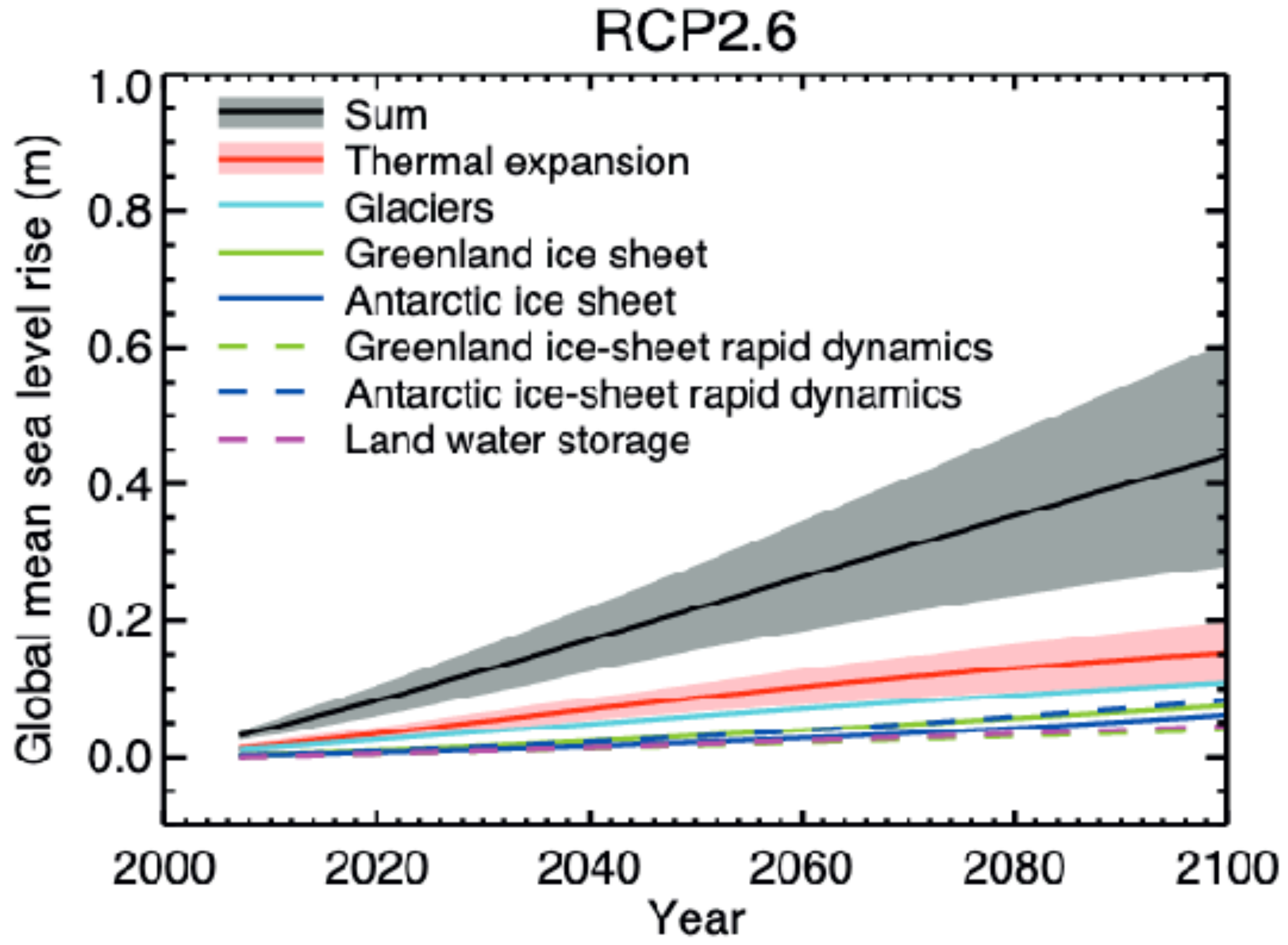
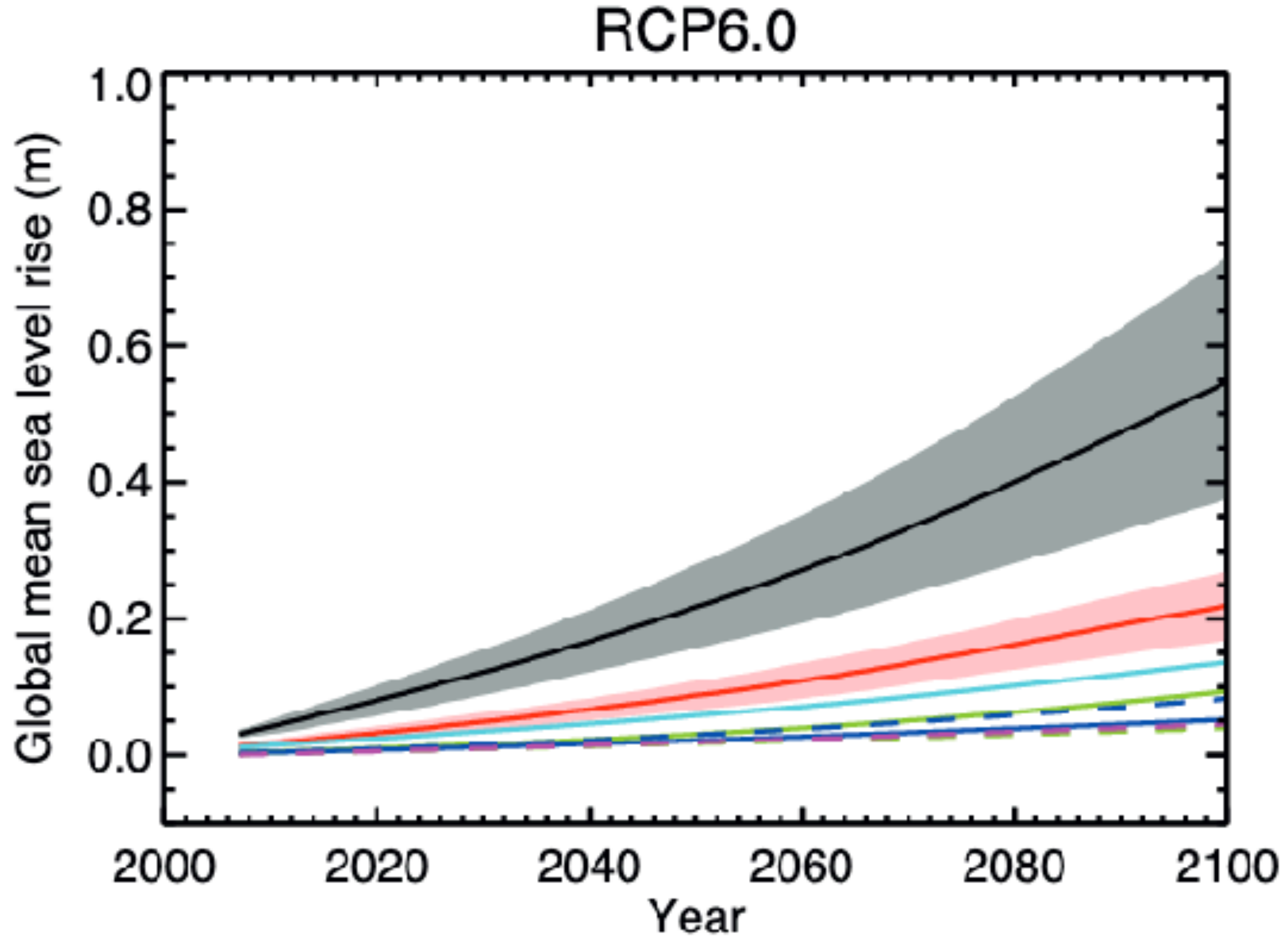
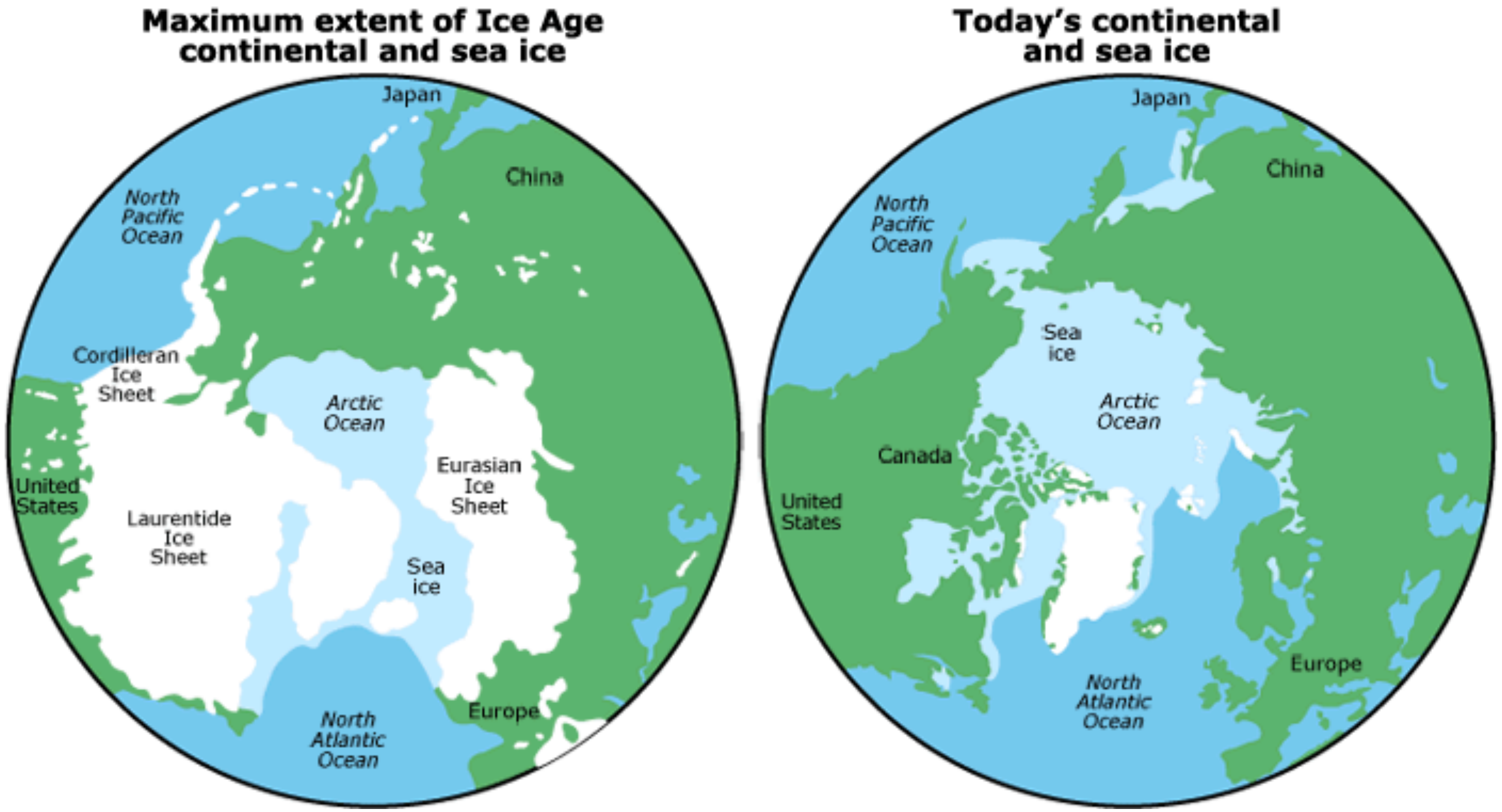
Glacial melt effects on sea-level rise

- Sea-level rise as a result of climate change is likely to be a major issue in the next century
- Of the contributions to sea-level rise, that due to glacial melt is expected to be the second largest
- Quantifying the uncertainty surrounding glacial effects on sea-level rise is crucial for decision making (governmental, private, personal, etc…)

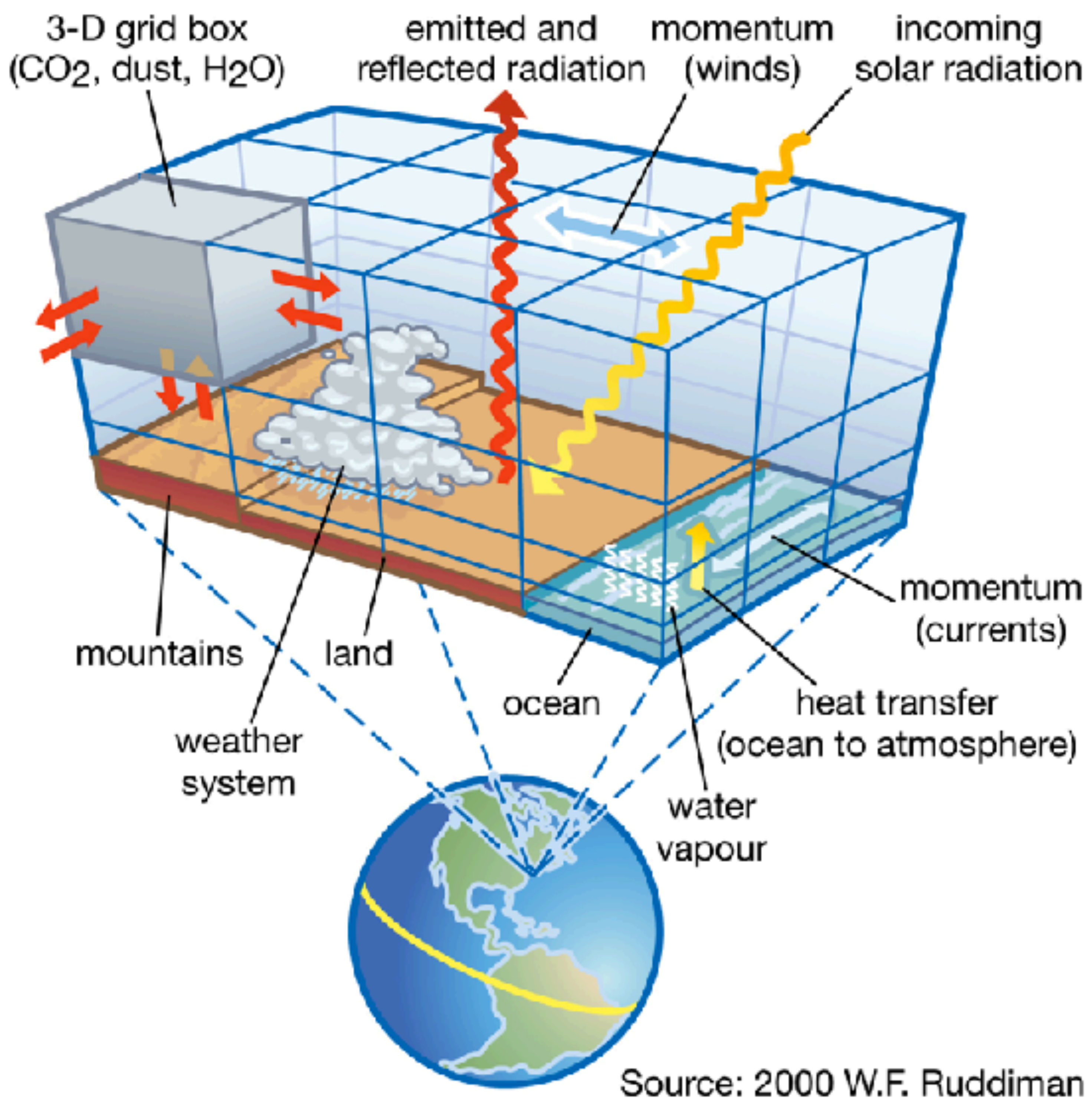




To understand glacial dynamics we look to the past



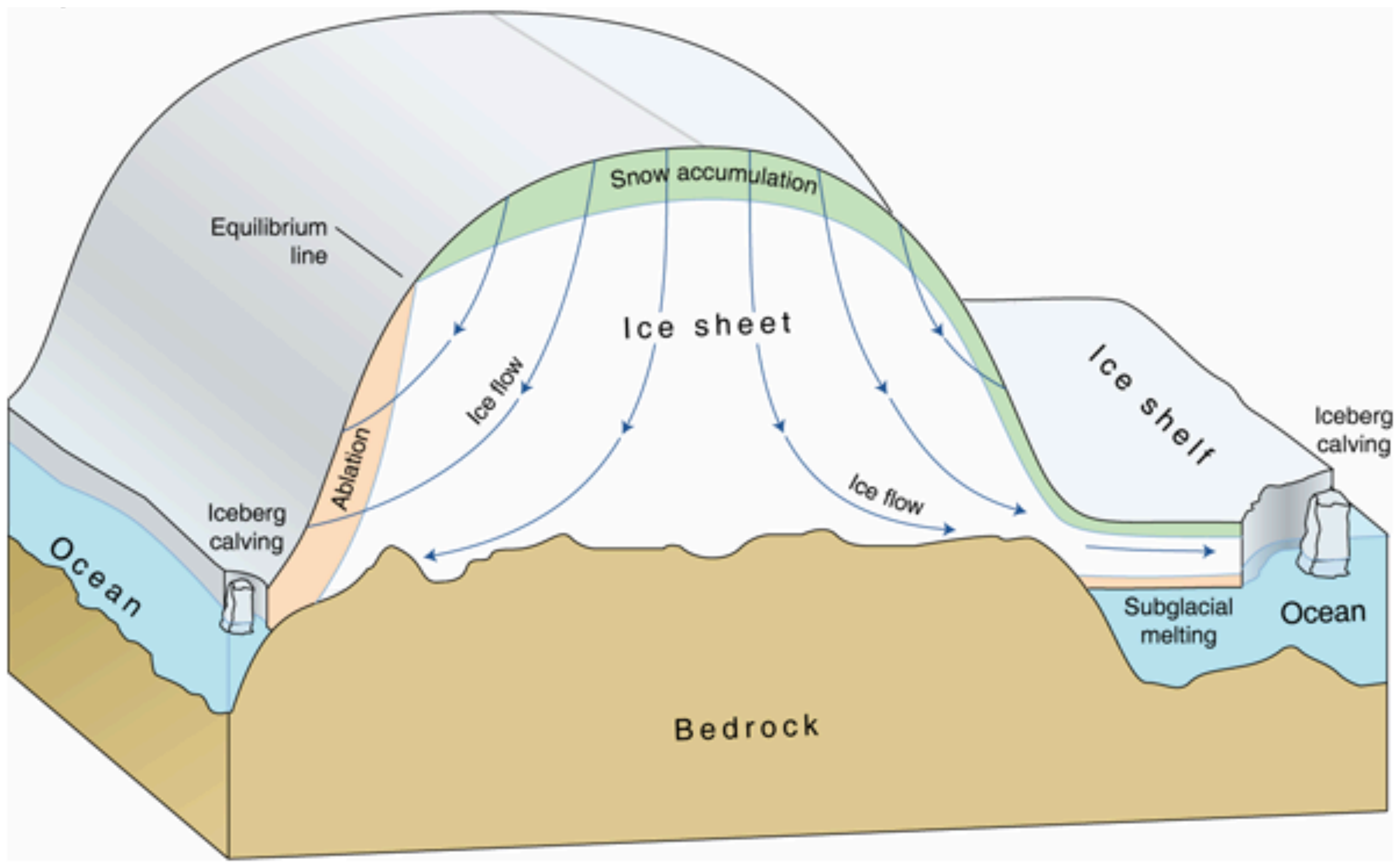
Modelling glacier dynamics is hard…



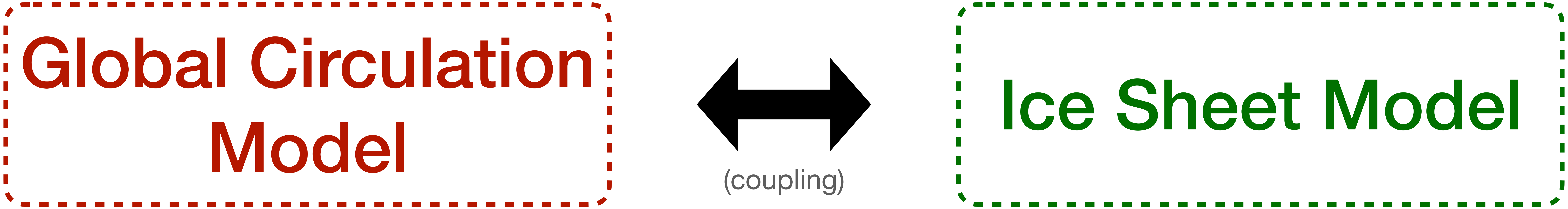
Global Circulation Models (GCM)

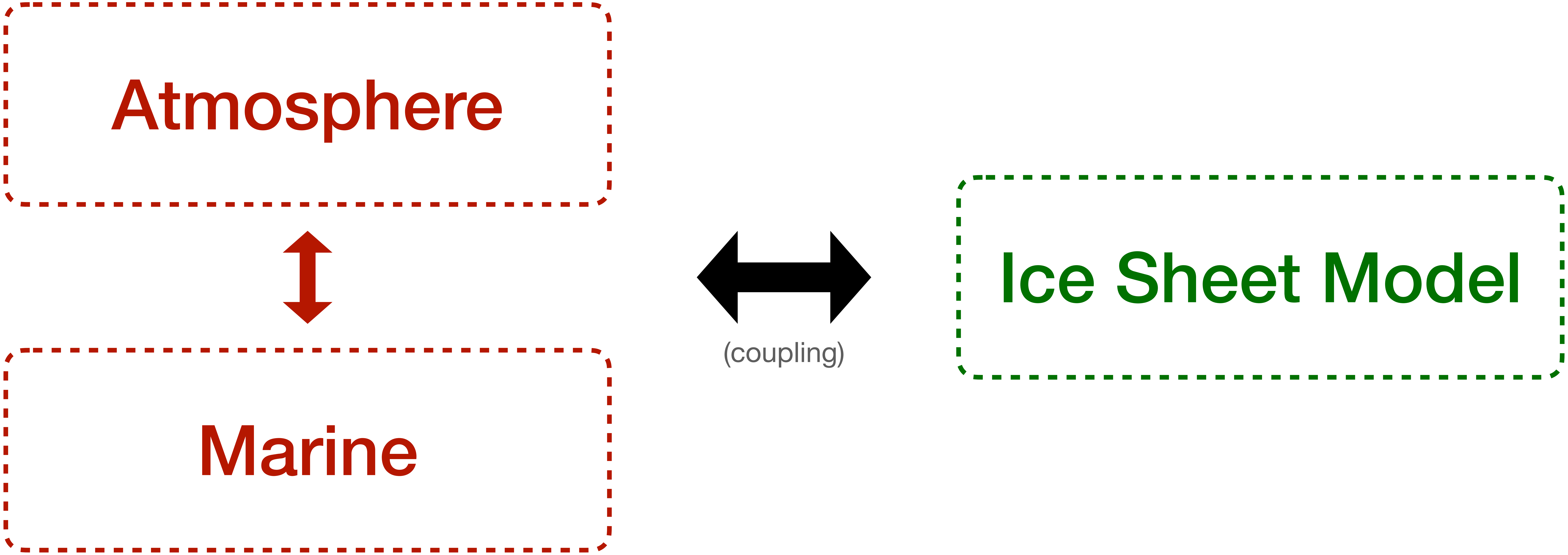
⟷

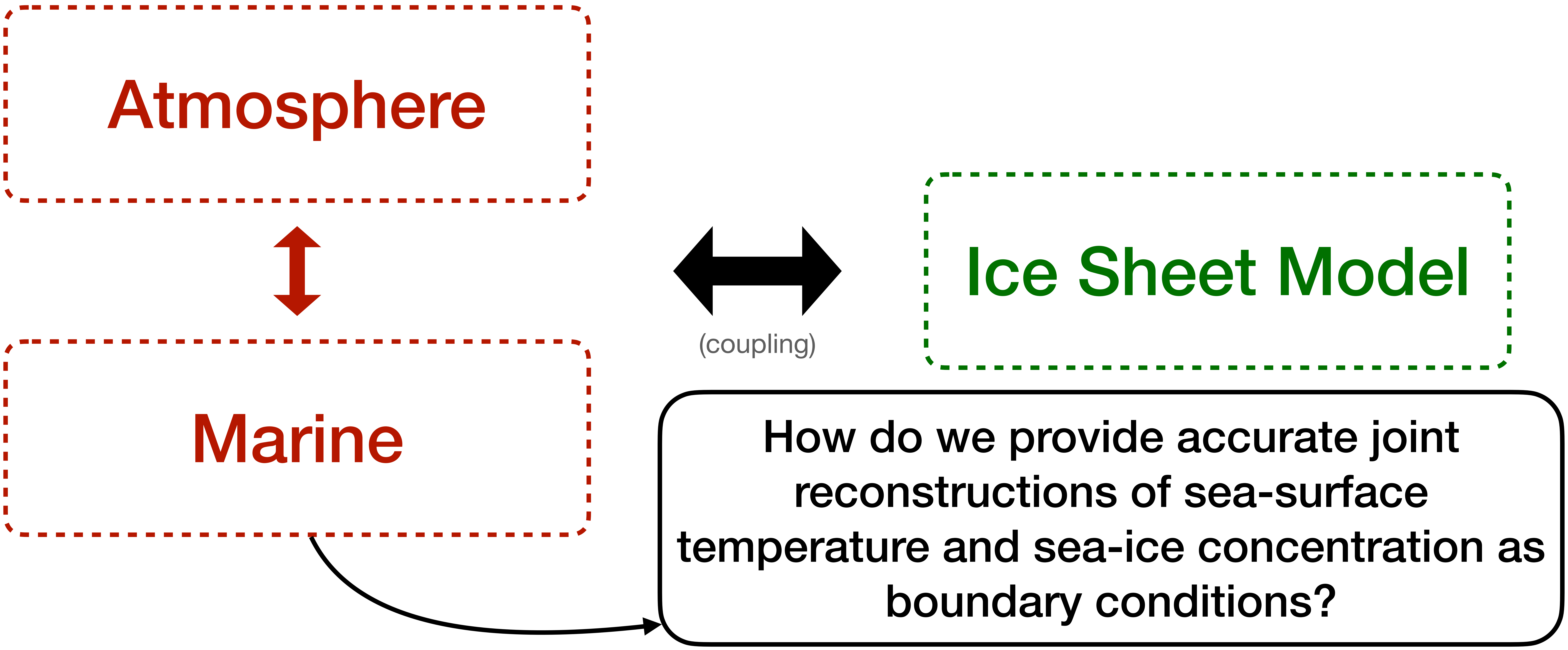
(coupling)



Regional Ice Sheet Models

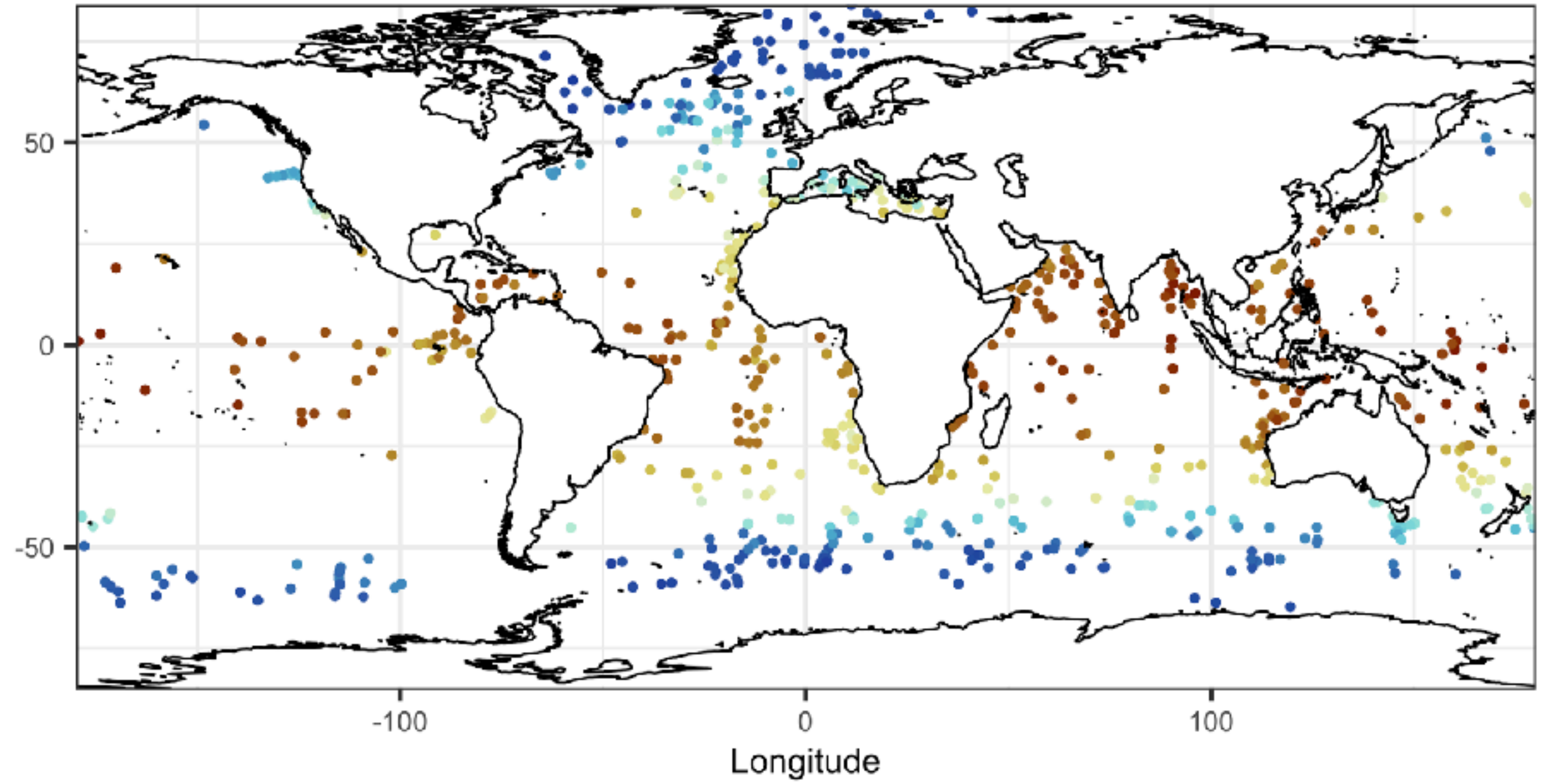

Global Circulation Model
(coupling)
Ice Sheet Model


Atmosphere
Marine
(coupling)
Ice Sheet Model

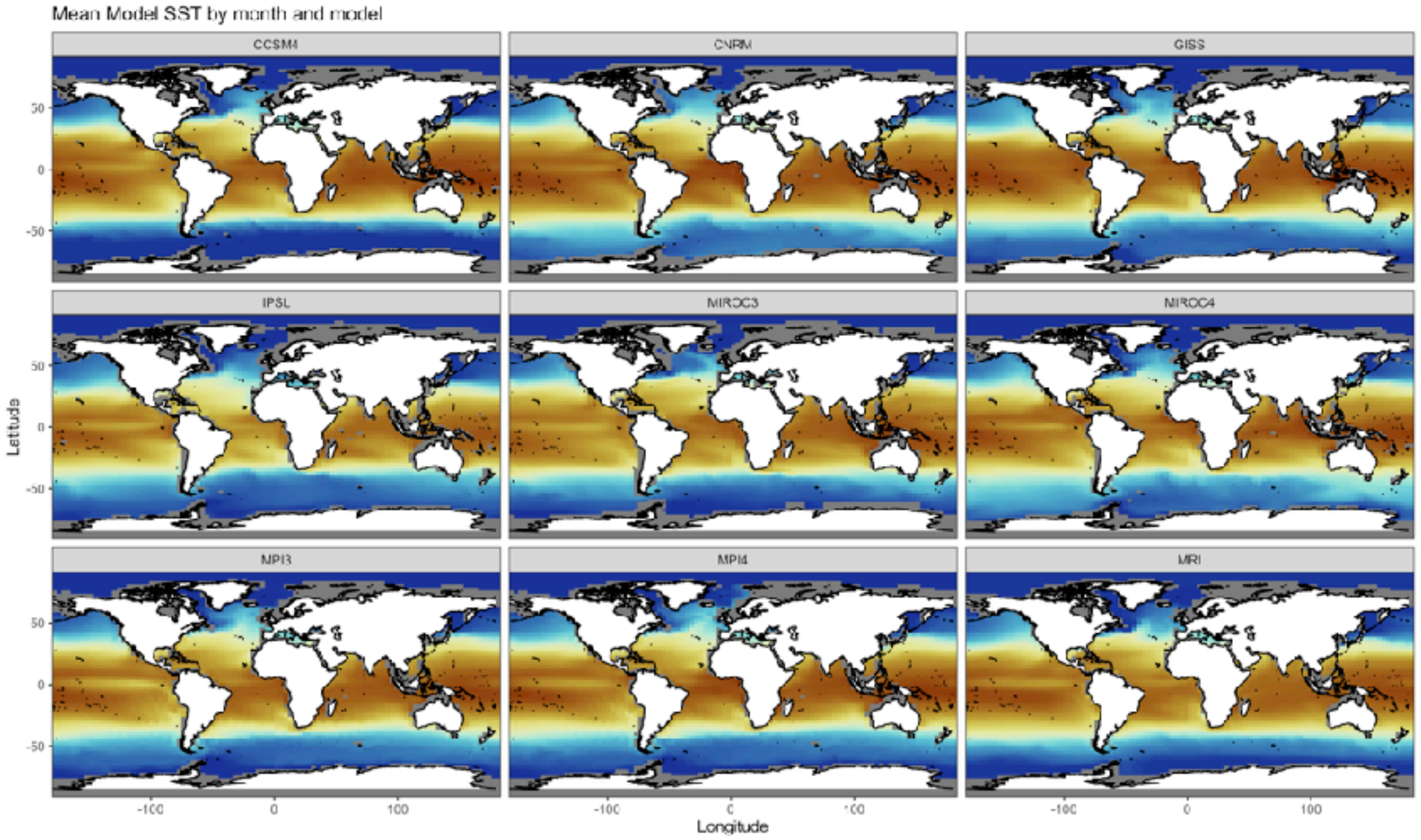

Atmosphere
Marine
(coupling)
Ice Sheet Model
How do we provide accurate joint reconstructions of sea-surface temperature and sea-ice concentration as boundary conditions?

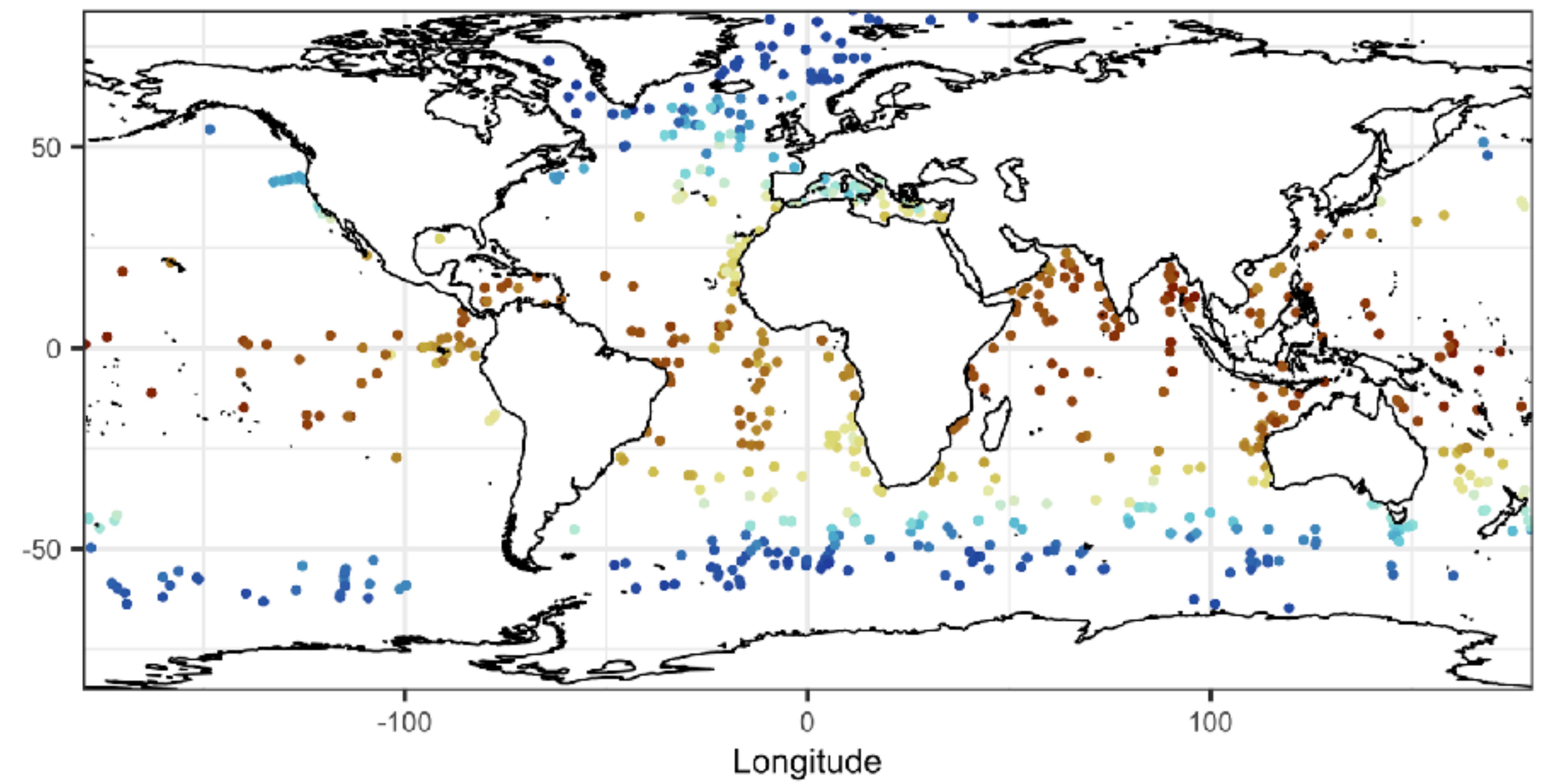
Data

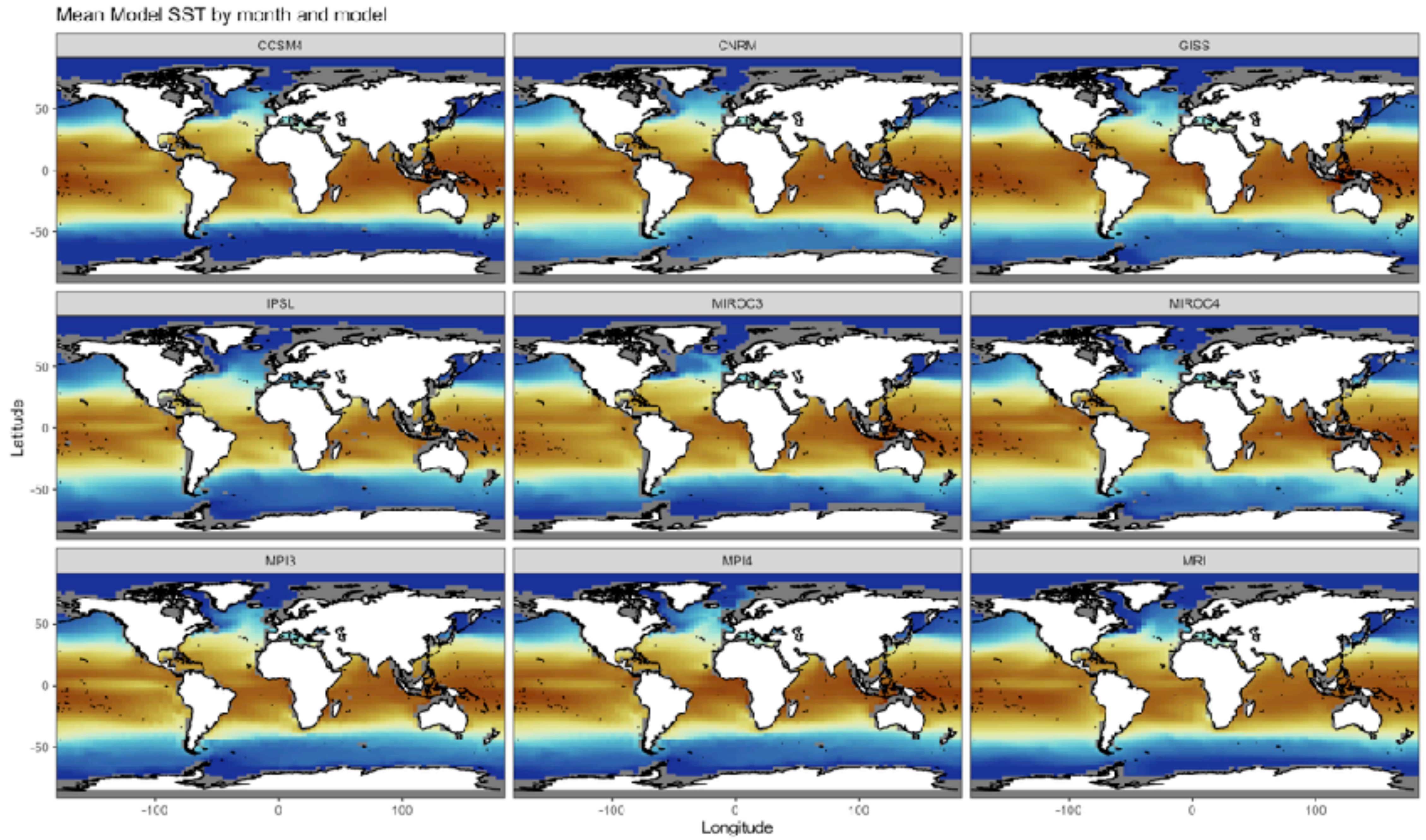
Model Runs

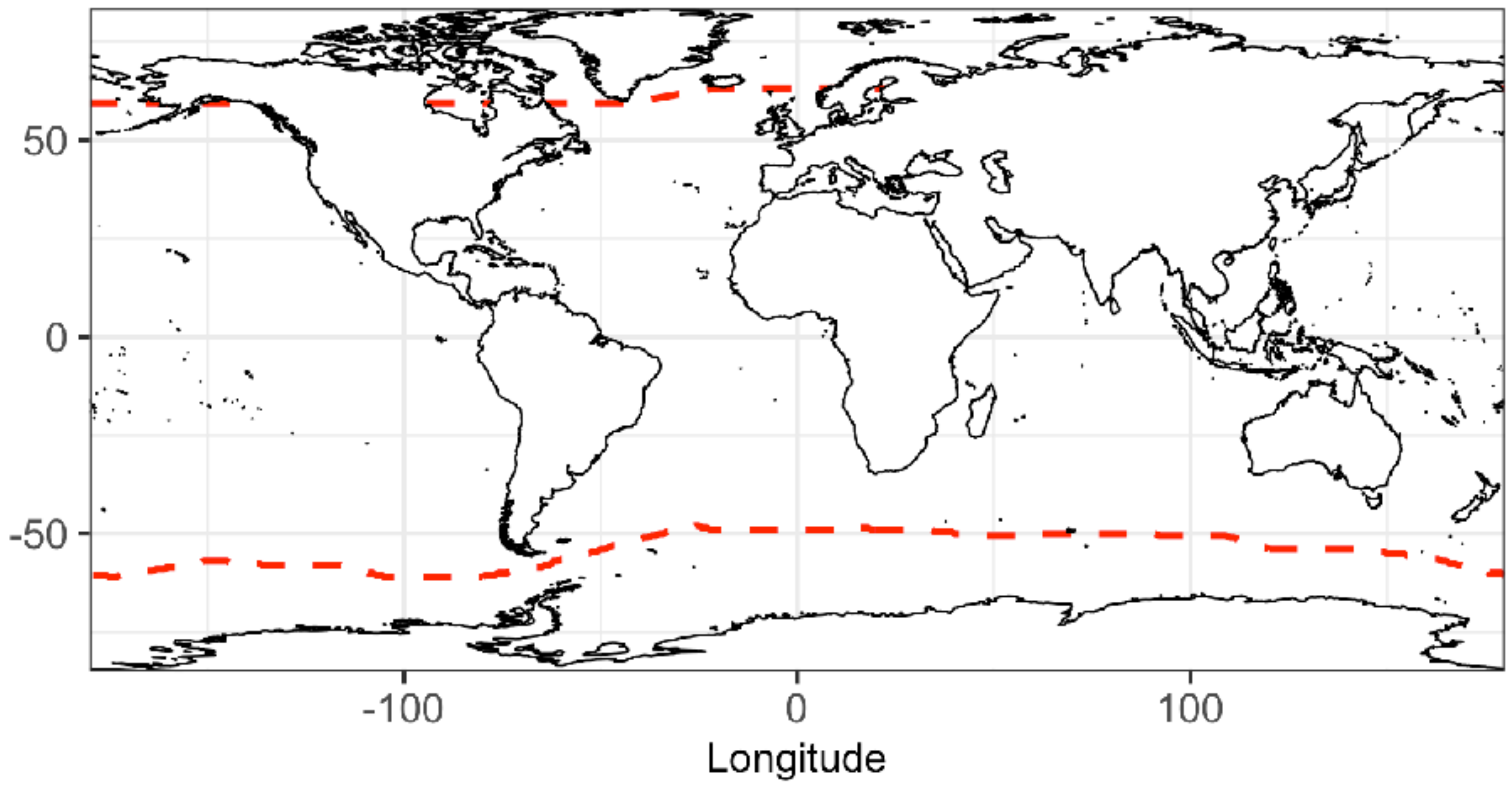


+




50
0
-50
-100
0
100
Longitude


Mean Model SST by month and model
CCSM4
CNRM
GISS
IPSL
MIROC3
MIROC4
MPI3
MPI4
MRI
Latitude
Longitude


50
0
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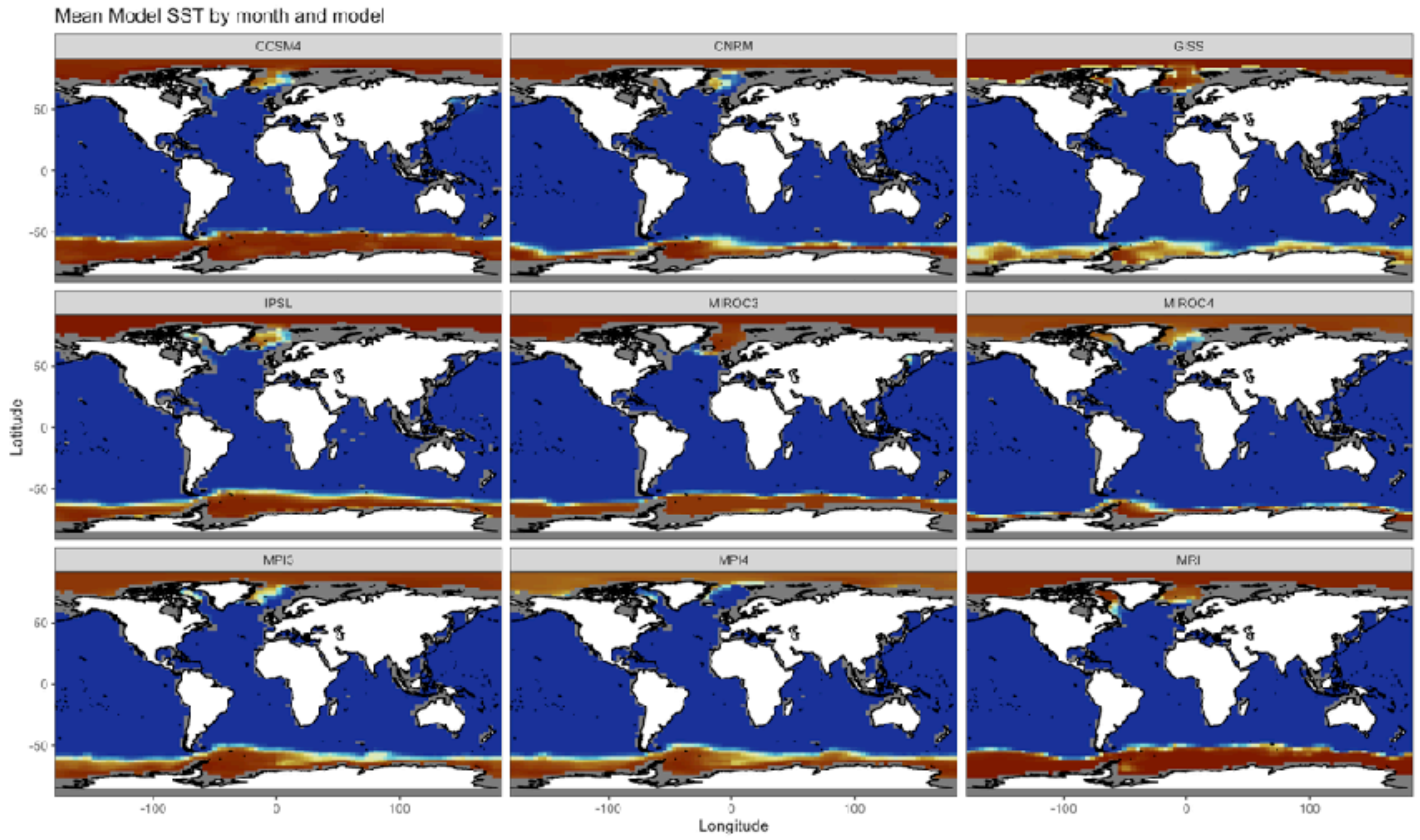
100
Longitude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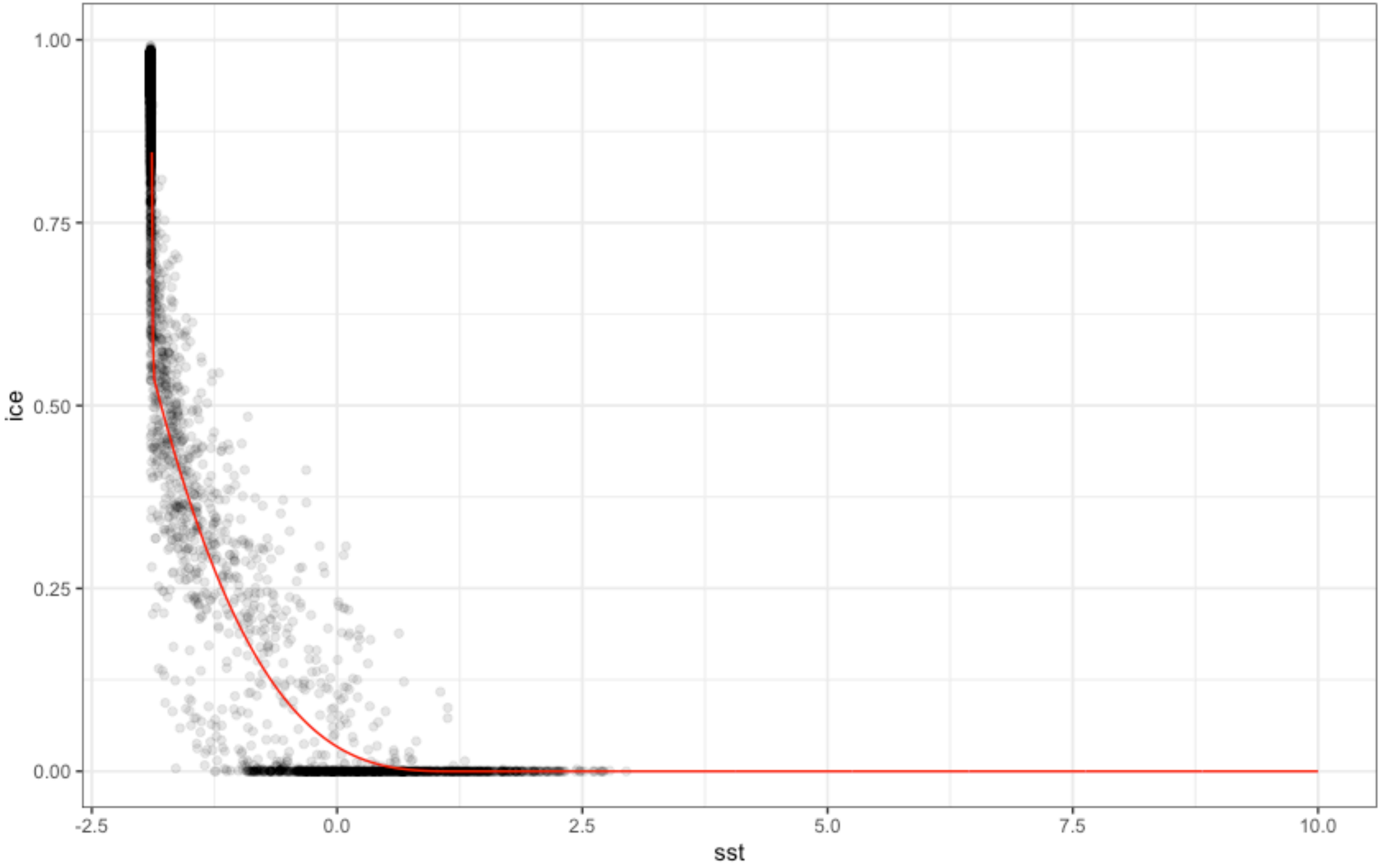
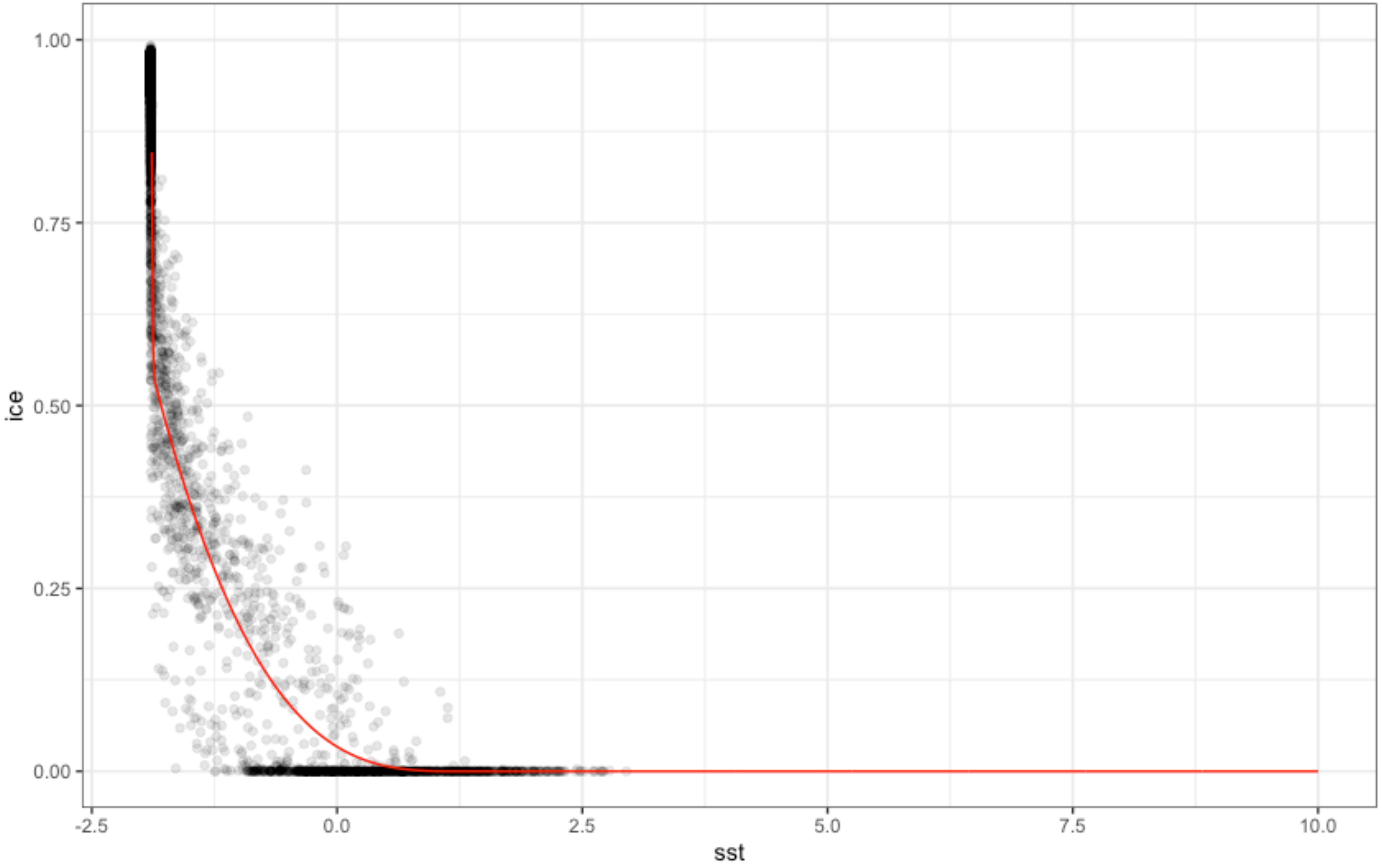
Joint behaviour of SST and SIC



Joint behaviour of SST and SIC



We have this at every grid cell in the model

Reconstructing boundary conditions

When forcing a climate model with boundary conditions the boundary conditions must

- be fully specified spatially and temporally,
- respect any physically related processes, and
- accurately represent the true process.

The coexchangeability model

The coexchangeability model

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

We assume exchangeability over the simulations,
i.e., cov$(X_i, X_j) = \Sigma \; \forall i, j$

The coexchangeability model

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

$$\mathbf{T}_{\mathbf{X}} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_{\mathbf{X}}$$

We assume coexchangeability between the simulations and the true process, i.e., cov$(X_i, T_X) = \Gamma \ \forall i$

The coexchangeability model

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

$$\mathbf{T}_{\mathbf{X}} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_{\mathbf{X}}$$

$$\mathbf{Z} = \mathbf{H}\mathbf{T}_{\mathbf{X}} + \mathbf{W}$$

We assume the data to be observed from the true latent process subject to some measurement error.

The coexchangeability model

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

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$$\mathbf{Z} = \mathbf{H}\mathbf{T}_{\mathbf{X}} + \mathbf{W}$$

This is the ‘coexchangeability model’ of Rougier et. al. (2013)

Co-adjusting sea-ice

- We do not have reliable measurements of SIC. Our best sea-ice data are maximum extents.
- We have joint simulations of SST and SIC, (x_i, y_i), that we can use to build a functional model of SIC given SST as an input; i.e. modelling $Y(X)$.
- From the model for X and the model for $Y(X)$ we build a model for Y.

Modelling SIC via exchangeable regressions

$$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$$

$$\beta_i = \mathcal{M}(\beta) + \mathcal{R}_i(\beta)$$

We assume conditional exchangeability over the simulations, i.e., cov$(Y_i, Y_j \mid X) = \Sigma \ \forall i, j$ and exchangeability over the parameters

Modelling SIC via exchangeable regressions

$$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$$

$$\beta_i = \mathcal{M}(\beta) + \mathcal{R}_i(\beta)$$

$$\mathbf{T_Y} = \Phi_{\mathbf{T_X}}\mathcal{M}(\beta) + \mathbf{U_Y}$$

We assume coexchangeability of the simulation parameters with the true process, i.e., cov($\beta_i, T_Y \mid X) = \Gamma \; \forall i, j$

The statistical model

SST	SIC
$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$	$$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$$ $$\beta_i = \mathcal{M}(\beta) + \mathcal{R}_i(\beta)$$
$$\mathbf{T}_\mathbf{X} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_\mathbf{X}$$	$$\mathbf{T}_\mathbf{Y} = \Phi_{\mathbf{T}_\mathbf{X}}\mathcal{M}(\beta) + \mathbf{U}_\mathbf{Y}$$
$$\mathbf{Z} = \mathbf{H}\mathbf{T}_\mathbf{X} + \mathbf{W}$$	

The statistical model

SST	SIC
$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$	$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$
	$\beta_i = \mathcal{M}(\beta) + \mathcal{R}_i(\beta)$
$\mathbf{T}_{\mathbf{X}} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_{\mathbf{X}}$	$\mathbf{T}_{\mathbf{Y}} = \Phi_{\mathbf{T}_{\mathbf{X}}}\mathcal{M}(\beta) + \mathbf{U}_{\mathbf{Y}}$
$\mathbf{Z} = \mathbf{H}\mathbf{T}_{\mathbf{X}} + \mathbf{W}$	

Bayes linear analysis

Bayes linear statistics does not presuppose probability and so is a powerful methodology when higher order judgements are not well founded.

To perform a Bayes linear analysis we

- choose expectation as our primitive (probability may then be defined from expectations of indicator functions),
- construct a vector space between the random quantity and the data,
- endow this space with an inner product, and
- perform inference via orthogonal projection in the inner product space.

Bayes Linear Adjustment of Beliefs

- Consider random quantity $X \in \mathscr{X}$ and data $D \in \mathscr{D}$ that form a linear space $\mathscr{L} = \{\mathscr{X}, \mathscr{D}\}$ with inner product $\langle A, B \rangle = \mathbb{E}[A^\intercal B]$
- Our adjusted expectation solves the orthogonal projection of X onto the affine space $\mathscr{A} = \{1, \mathscr{D}\}$, i.e., for $\mathbb{E}_D[X] = h1 + HD$, $\mathbb{E}_D[X] = \arg\min_{\mathbb{E}_D[X]} \|(X - \mathbb{E}_D[X])\|^2$ with solution

$$\mathbb{E}_D[X] = \mathbb{E}[X] + \mathrm{cov}[X, D]\mathrm{var}[D]^+(D - \mathbb{E}[D])$$

- The adjusted variance, $\mathrm{var}_D[X]$, is the outer product $\mathbb{E}[(X - \mathbb{E}_D[X])(X - \mathbb{E}_D[X])^\intercal]$, and is

$$\mathrm{var}_D[X] = \mathrm{var}[X] - \mathrm{cov}[X, D]\mathrm{var}[D]^+\mathrm{cov}[D, X]$$

Inference

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

$$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$$

$$\beta_i = \mathcal{M}(\beta) + \mathcal{R}_i(\beta)$$

$$\mathbf{T}_{\mathbf{X}} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_{\mathbf{X}}$$

$$\mathbf{T}_{\mathbf{Y}} = \Phi_{\mathbf{T}_{\mathbf{X}}}\mathcal{M}(\beta) + \mathbf{U}_{\mathbf{Y}}$$

$$\mathbf{Z} = \mathbf{H}\mathbf{T}_{\mathbf{X}} + \mathbf{W}$$

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$$\mathbf{Z} = \mathbf{H}\mathbf{T}_{\mathbf{X}} + \mathbf{W}$$

1. BL update by $\bar{X}$ of T_X to calculate $\mathbb{E}_{\bar{X}}[T_X]$ and var$_{\bar{X}}[T_X]$

Inference

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

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2. BL update by Z of updated T_X to calculate $\mathbb{E}_{\bar{X},Z}[T_X]$ and $\text{var}_{\bar{X},Z}[T_X]$

Inference

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

$$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$$

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$$\mathbf{T}_\mathbf{X} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_\mathbf{X}$$

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3. BL update $M(\beta)$ by $(x_i, y_i)\ldots$ to calculate $\mathbb{E}_{(X,Y)}[M(\beta)]$ and $\text{var}_{(X,Y)}[M(\beta)]$

Inference

$$\mathbf{X}_i = \mathcal{M}(\mathbf{X}) + \mathcal{R}_i(\mathbf{X})$$

$$\mathbf{Y}_i = \Phi_{\mathbf{X}_i}\beta_i + \epsilon_i$$

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$$\mathbf{T}_\mathbf{X} = \mathcal{M}(\mathbf{X}) + \mathbf{U}_\mathbf{X}$$

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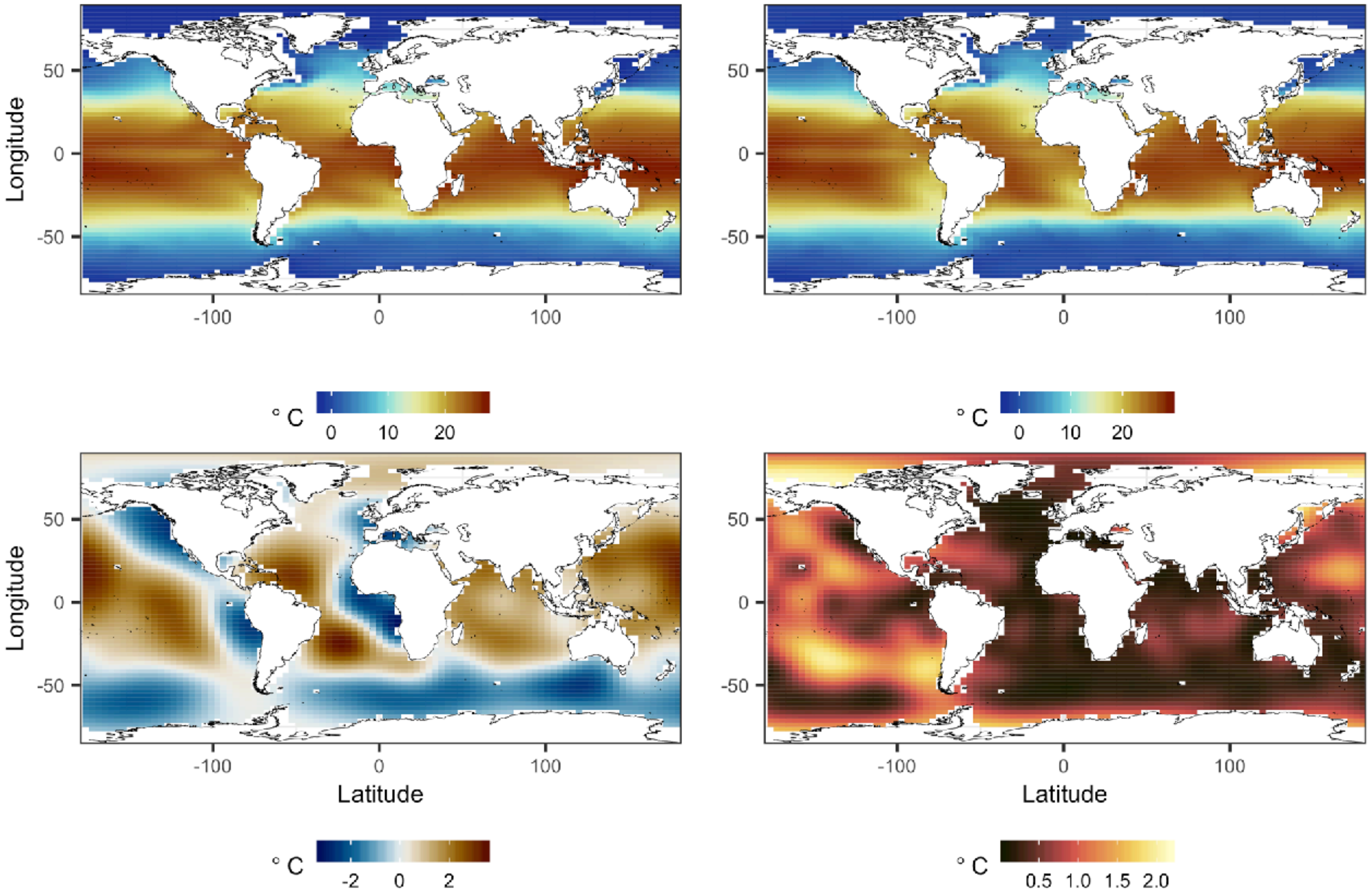
$$\mathbf{Z} = \mathbf{H}\mathbf{T}_\mathbf{X} + \mathbf{W}$$

1. BL update by $\bar{X}$ of T_X to calculate $\mathbb{E}_{\bar{X}}[T_X]$ and $\text{var}_{\bar{X}}[T_X]$
2. BL update by Z of updated T_X to calculate $\mathbb{E}_{\bar{X},Z}[T_X]$ and $\text{var}_{\bar{X},Z}[T_X]$
3. BL update $M(\beta)$ by $(x_i, y_i)\ldots$ to calculate $\mathbb{E}_{(X,Y)}[M(\beta)]$ and $\text{var}_{(X,Y)}[M(\beta)]$
4. History match samples of T_X and T_Y to observed functions of T_Y to obtain joint plausibility samples

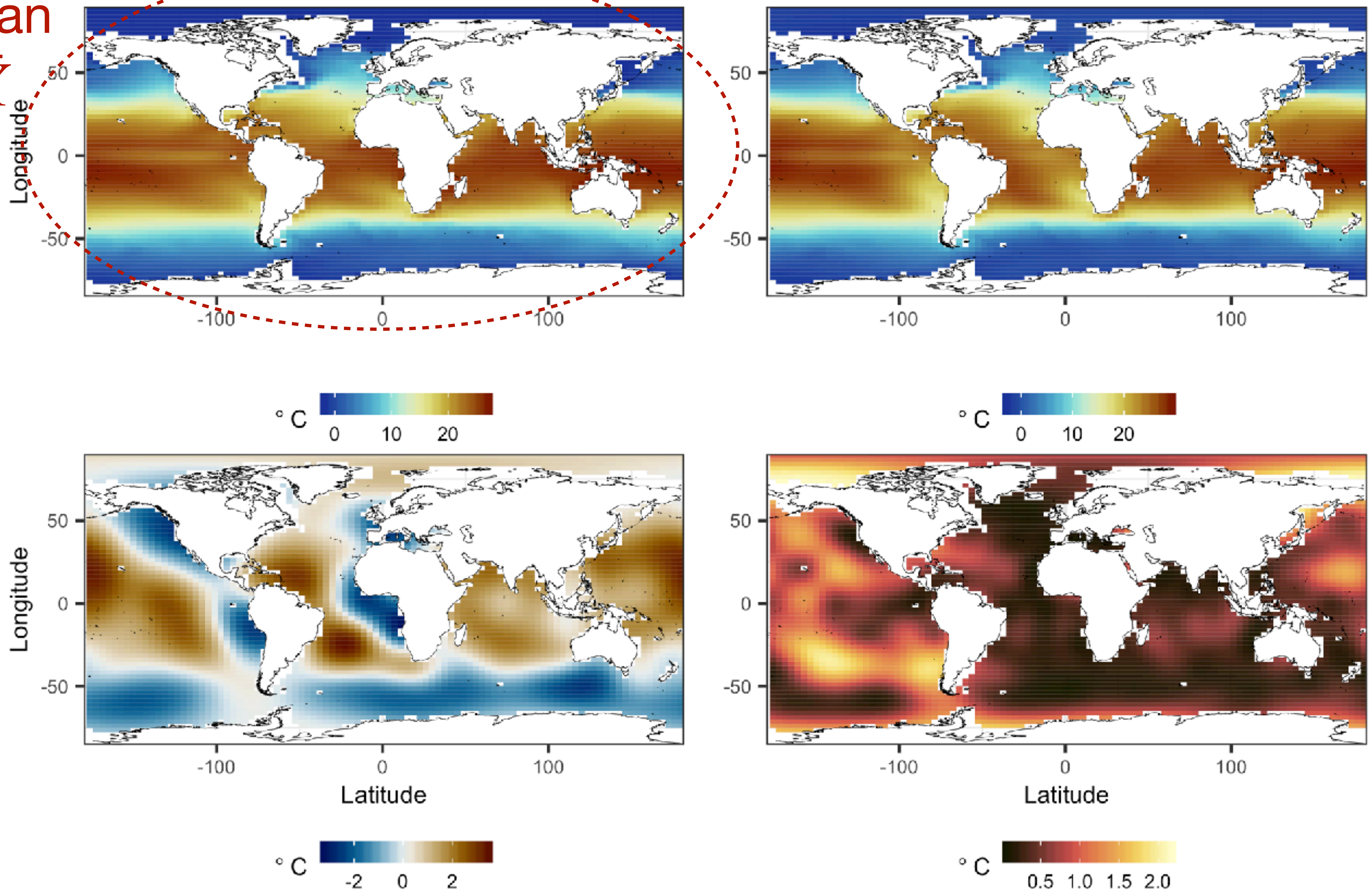
Adjusted beliefs of T_X



Adjusted beliefs of T_X

Ensemble Mean

$\mathbb{E}_{\bar{X}}[T_X] = \bar{X}$



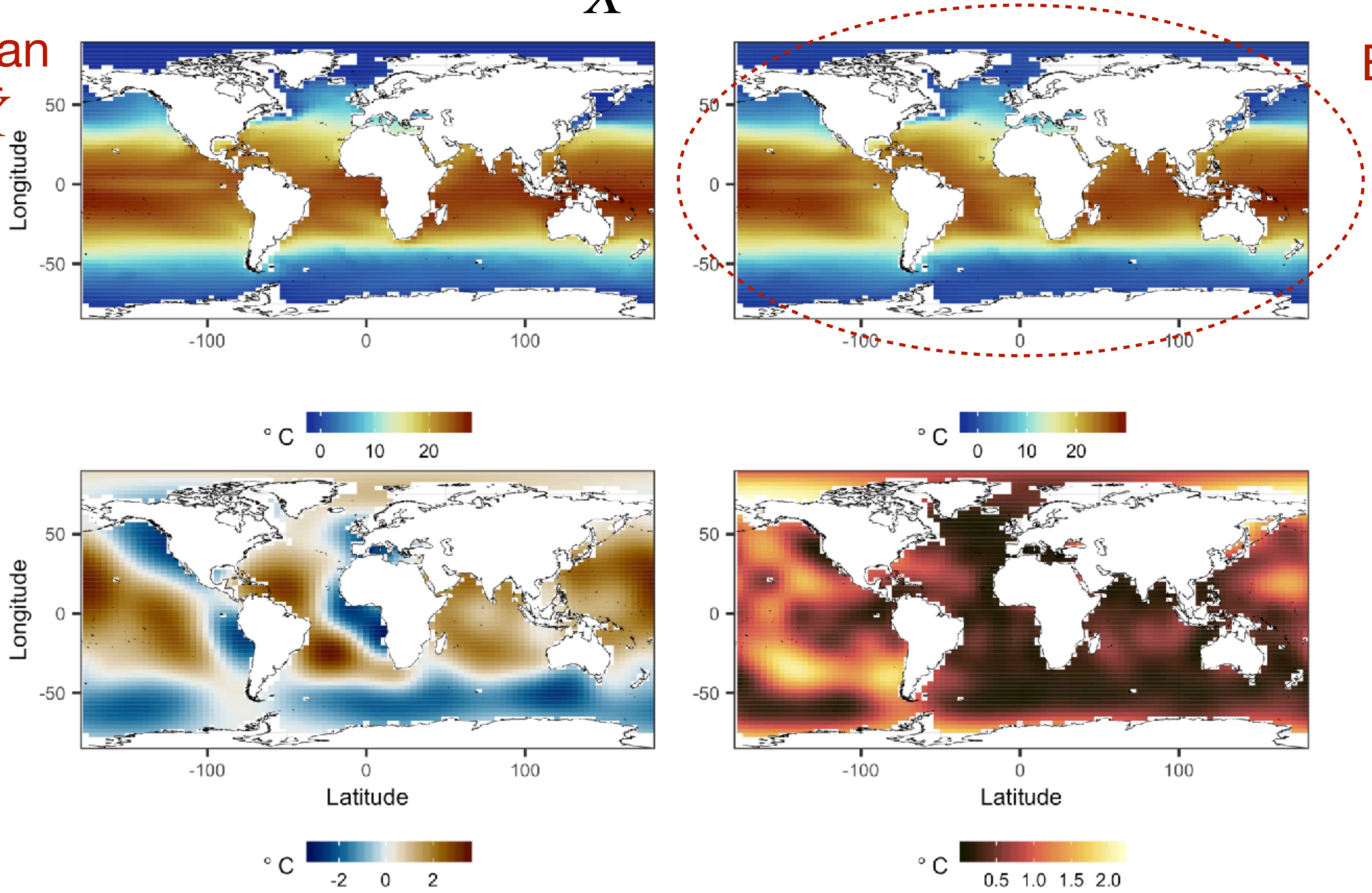
Adjusted beliefs of T_X

Ensemble Mean

$\mathbb{E}_{\bar{X}}[T_X] = \bar{X}$

Adjusted Expectation

$\mathbb{E}_{\bar{X},Z}[T_X]$

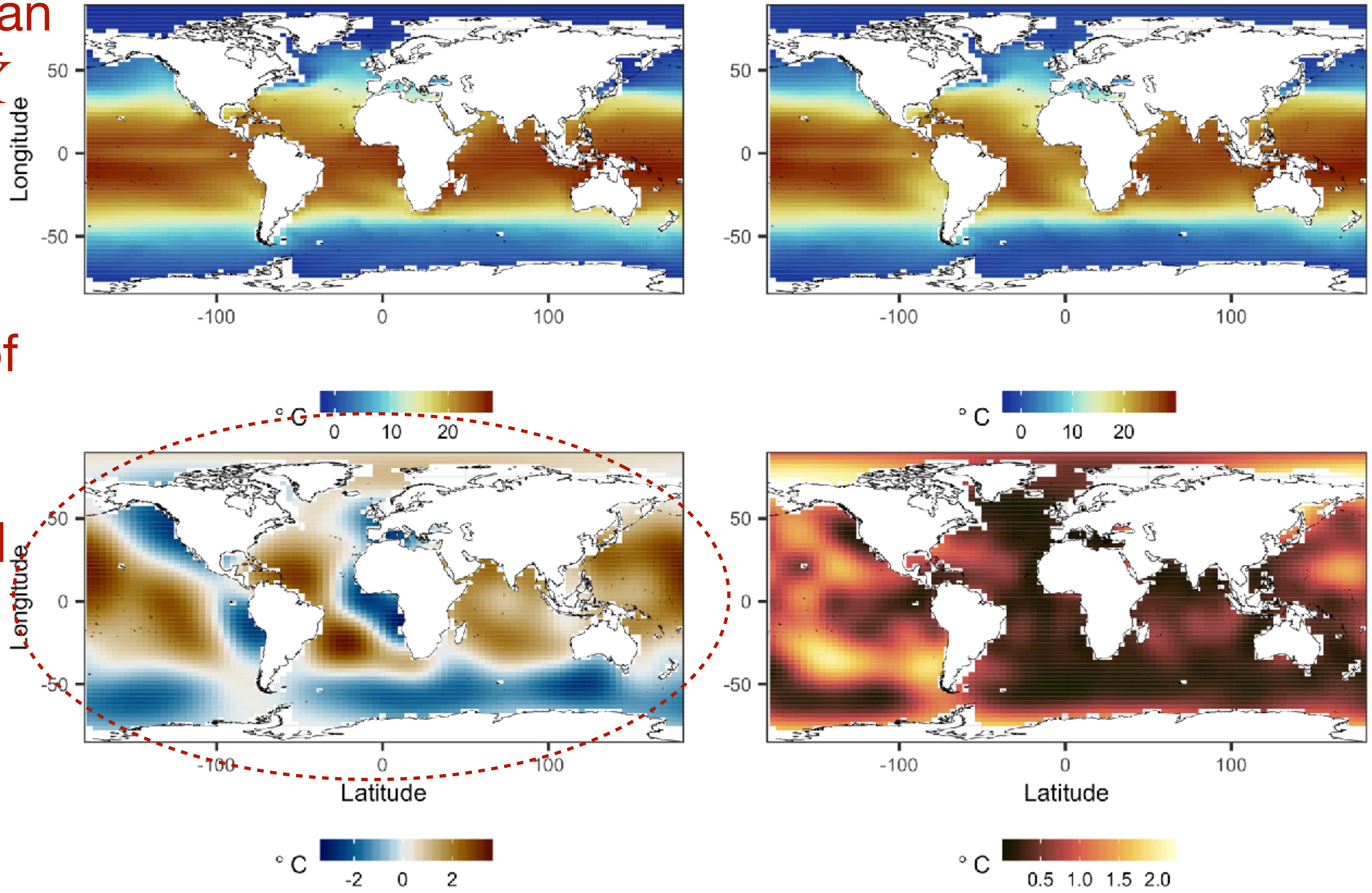


Adjusted beliefs of T_X

Ensemble Mean
$\mathbb{E}_{\bar{X}}[T_X] = \bar{X}$

Adjusted Expectation
$\mathbb{E}_{\bar{X},Z}[T_X]$

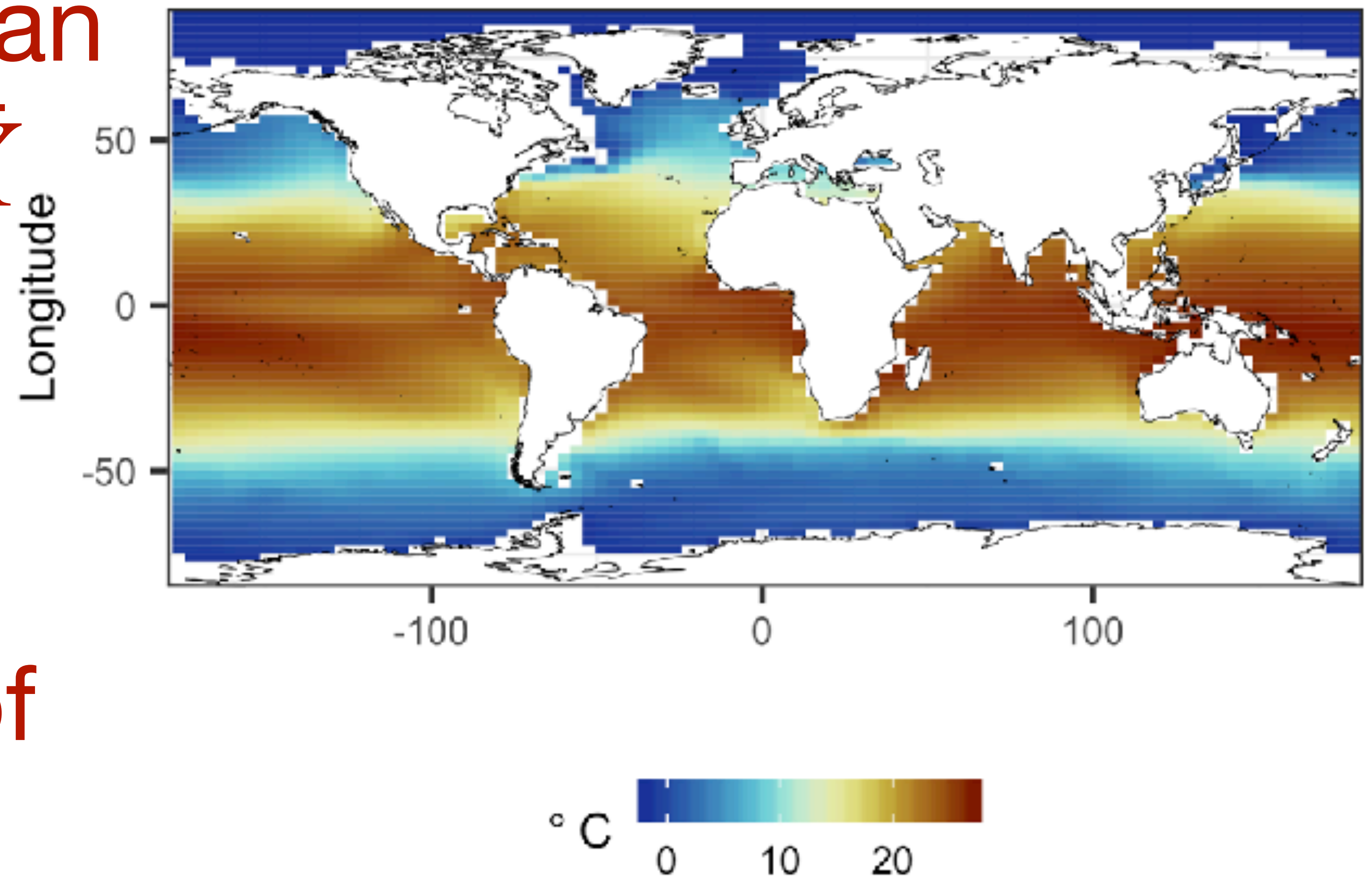
Contribution of data to adjustment
$\mathbb{E}_{\bar{X},Z}[T_X]$ - $\mathbb{E}_{\bar{X}}[T_X]$



Adjusted beliefs of T_X

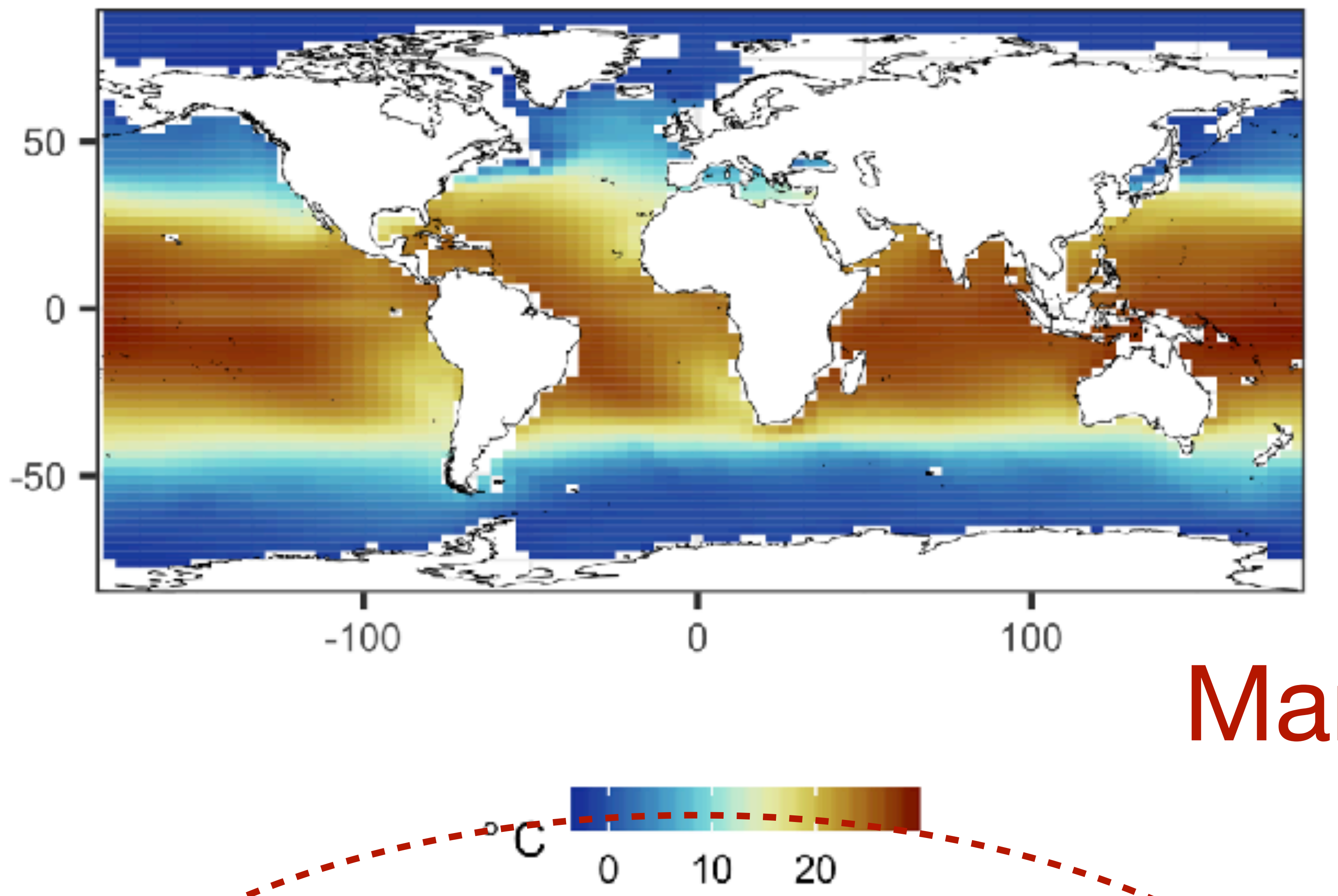
Ensemble Mean

$\mathbb{E}_{\bar{X}}[T_X] = \bar{X}$



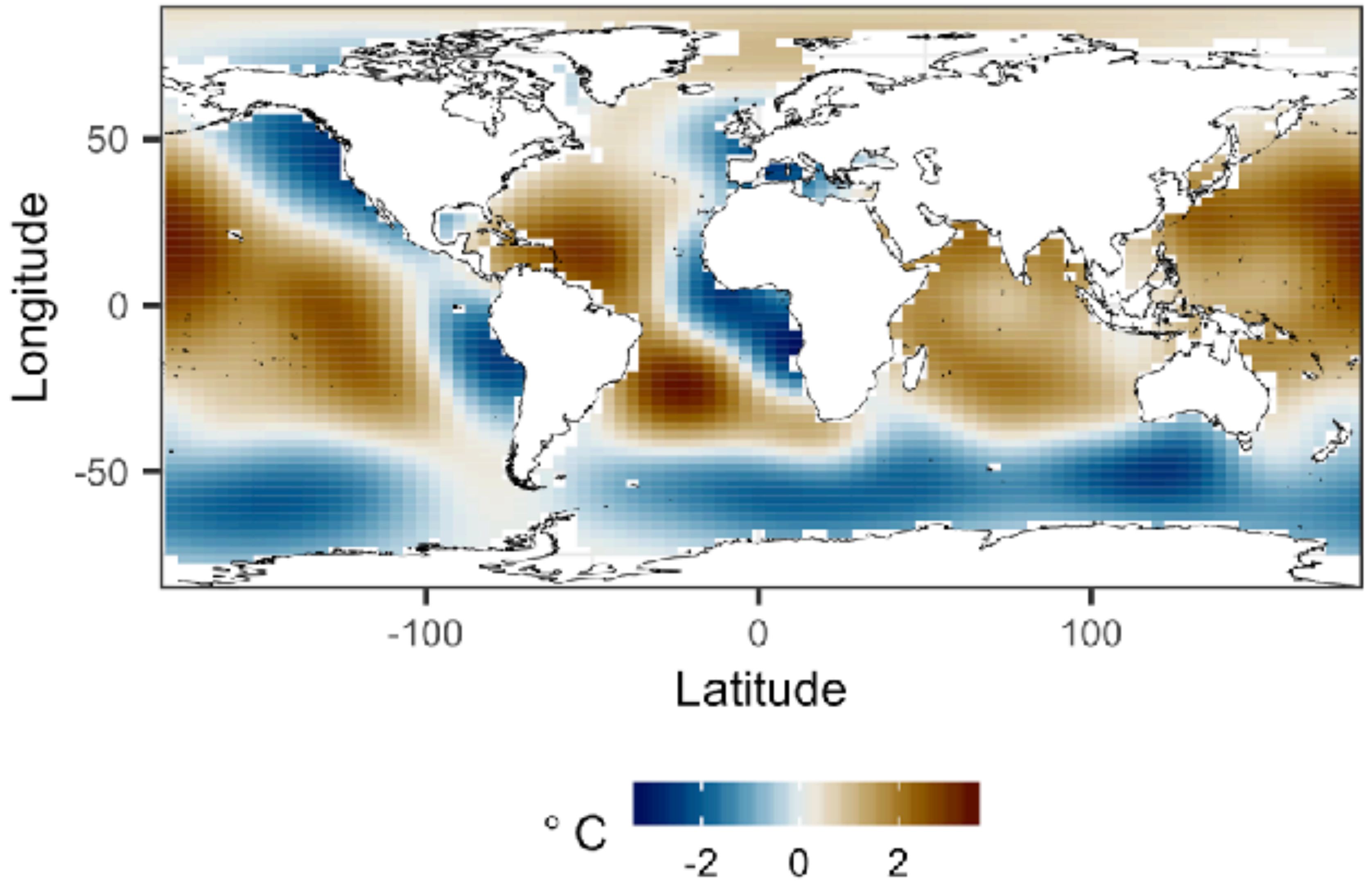
Adjusted Expectation

$\mathbb{E}_{\bar{X},Z}[T_X]$



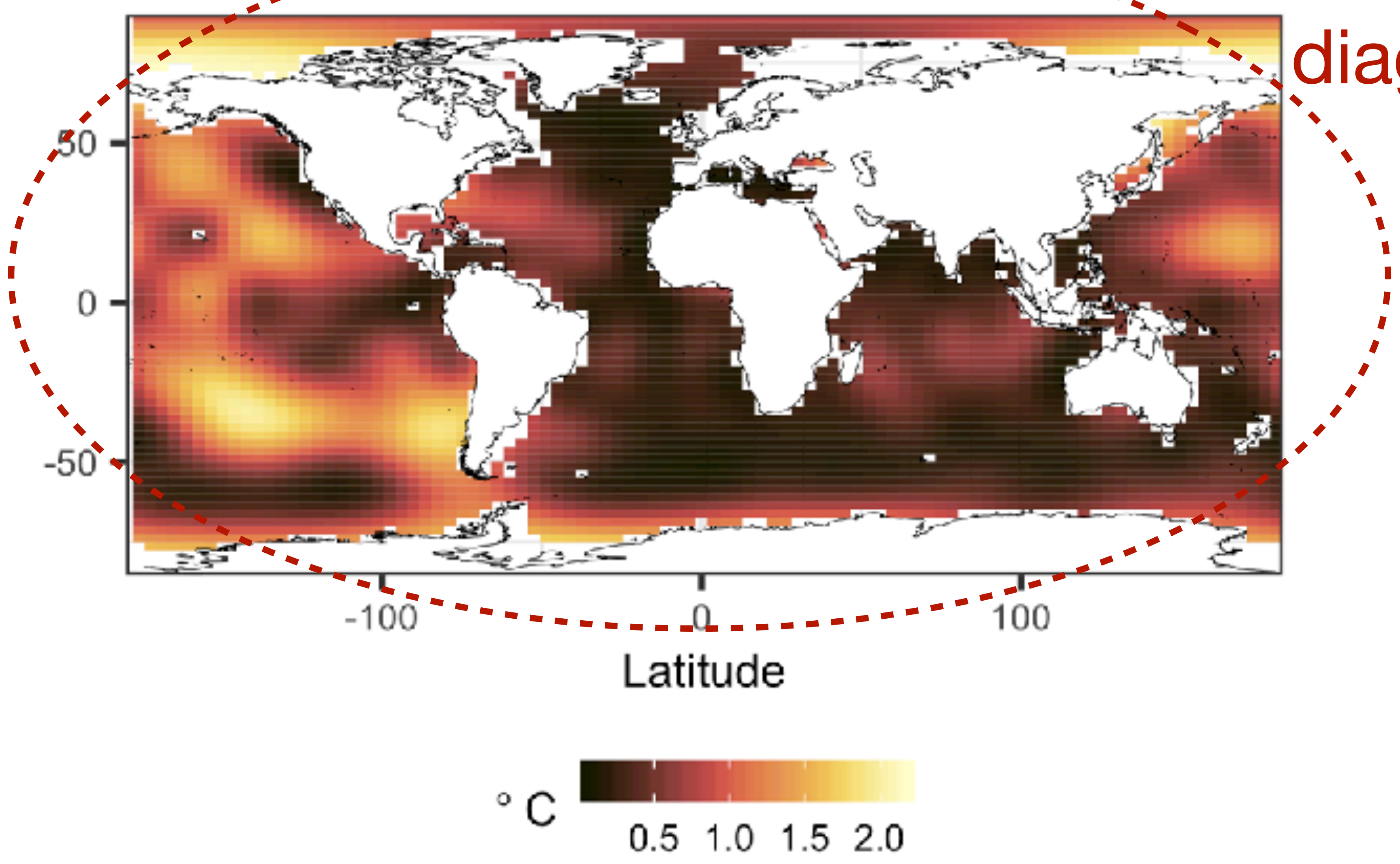
Contribution of data to adjustment

$\mathbb{E}_{\bar{X},Z}[T_X]$ - $\mathbb{E}_{\bar{X}}[T_X]$

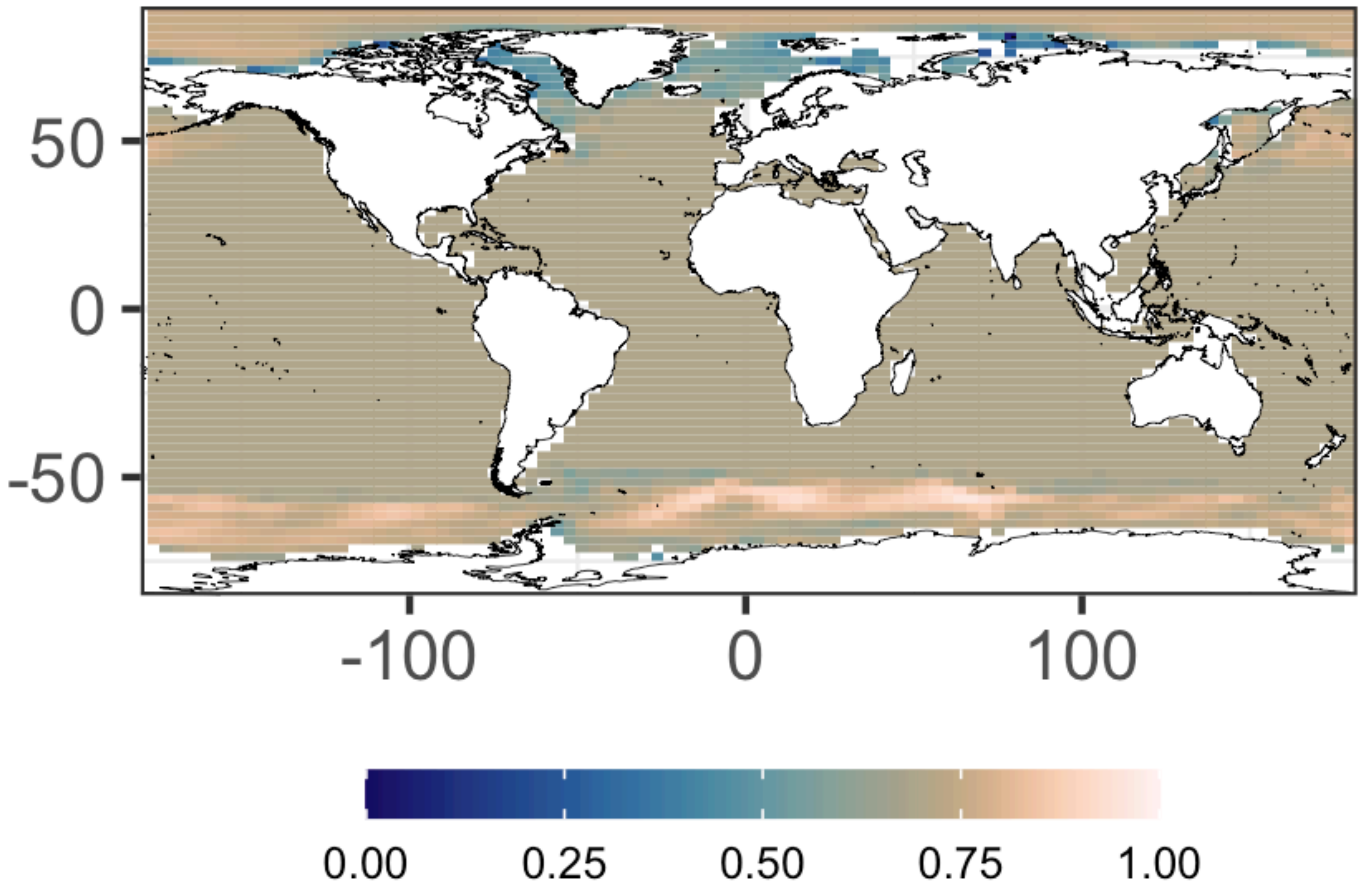


Marginal adjusted variance

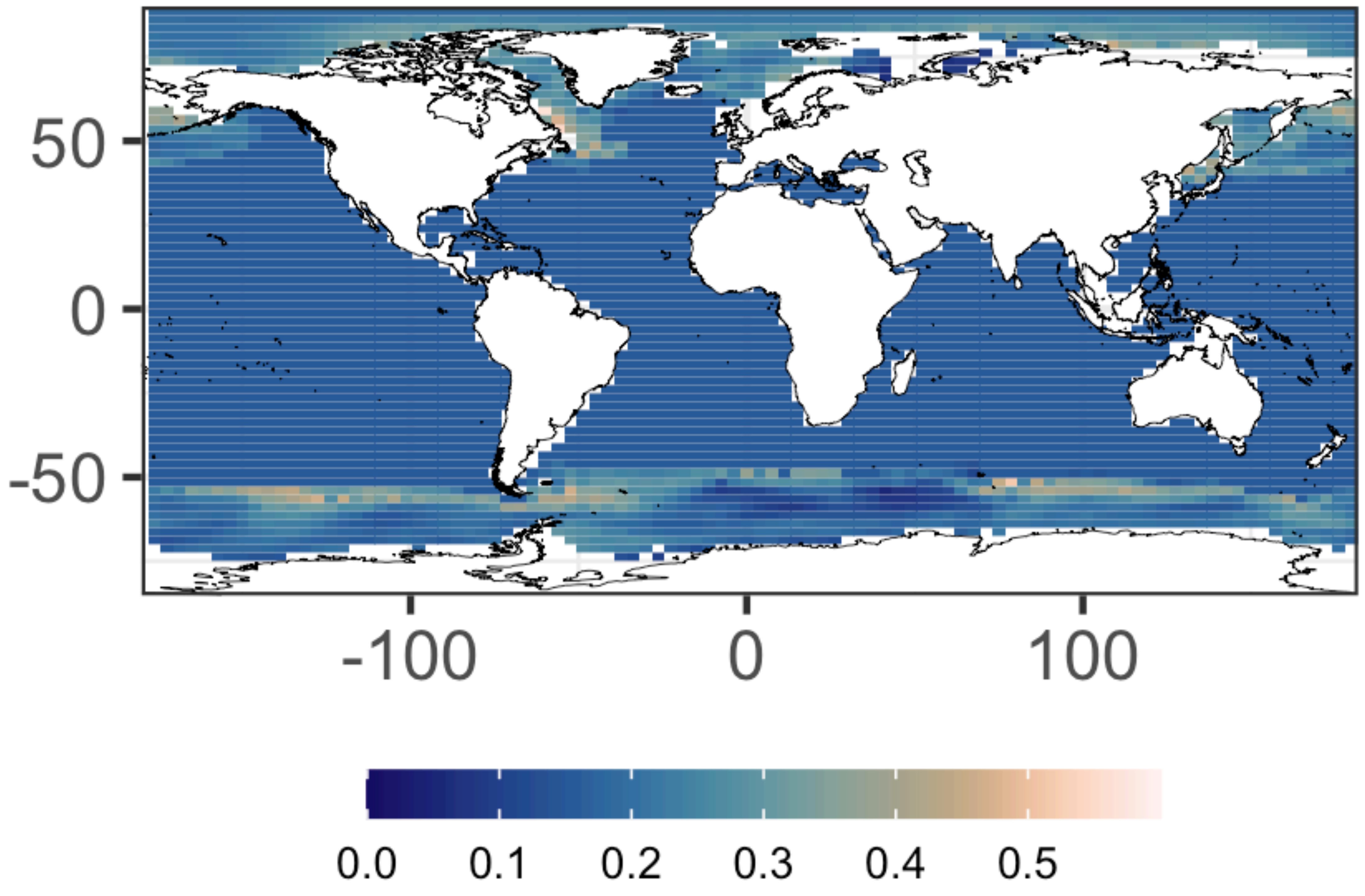
diag(var$_{\bar{X},Z}[T_X]$)



Some adjusted beliefs of $M(\beta)$



$\mathbb{E}_{(X,Y)}[M(\beta_2)]$

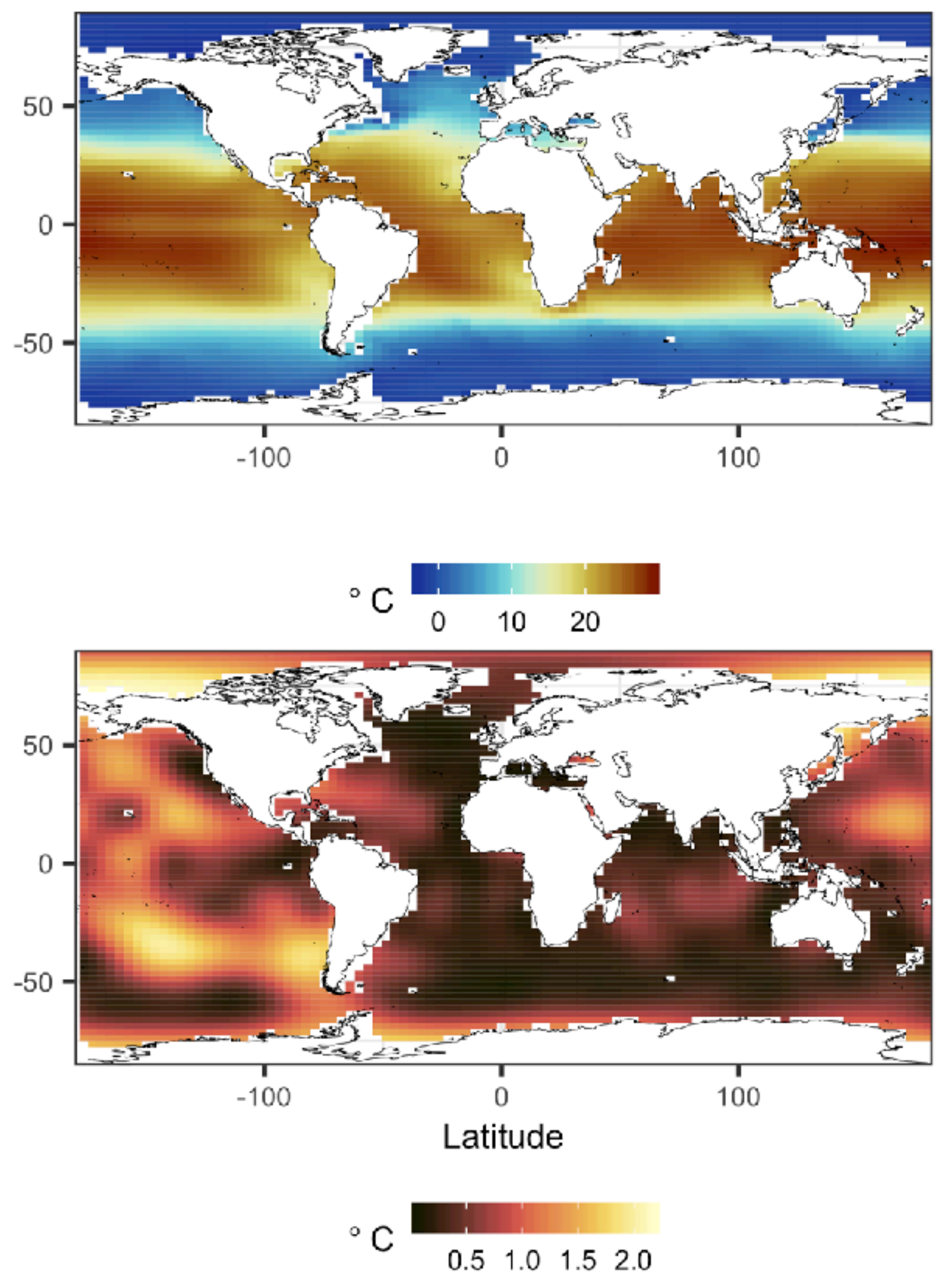


$\mathbb{E}_{(X,Y)}[M(\beta_3)]$

History Matching SIC

Generate

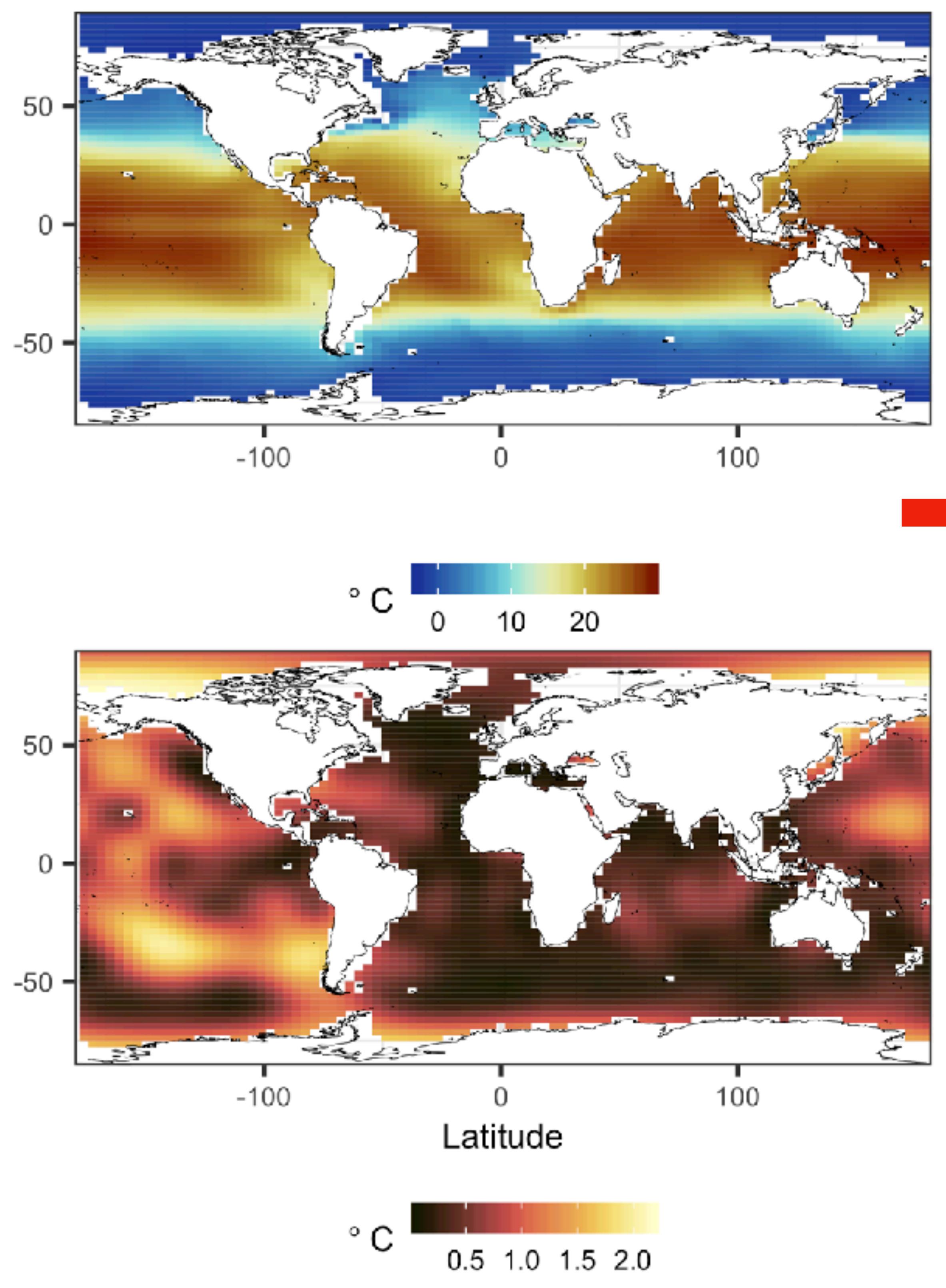
$$\tilde{T}_X \sim \mathbb{E}_{X,Z}[T_X] + (\text{var}_{X,Z}[T_X])^{1/2}\epsilon$$

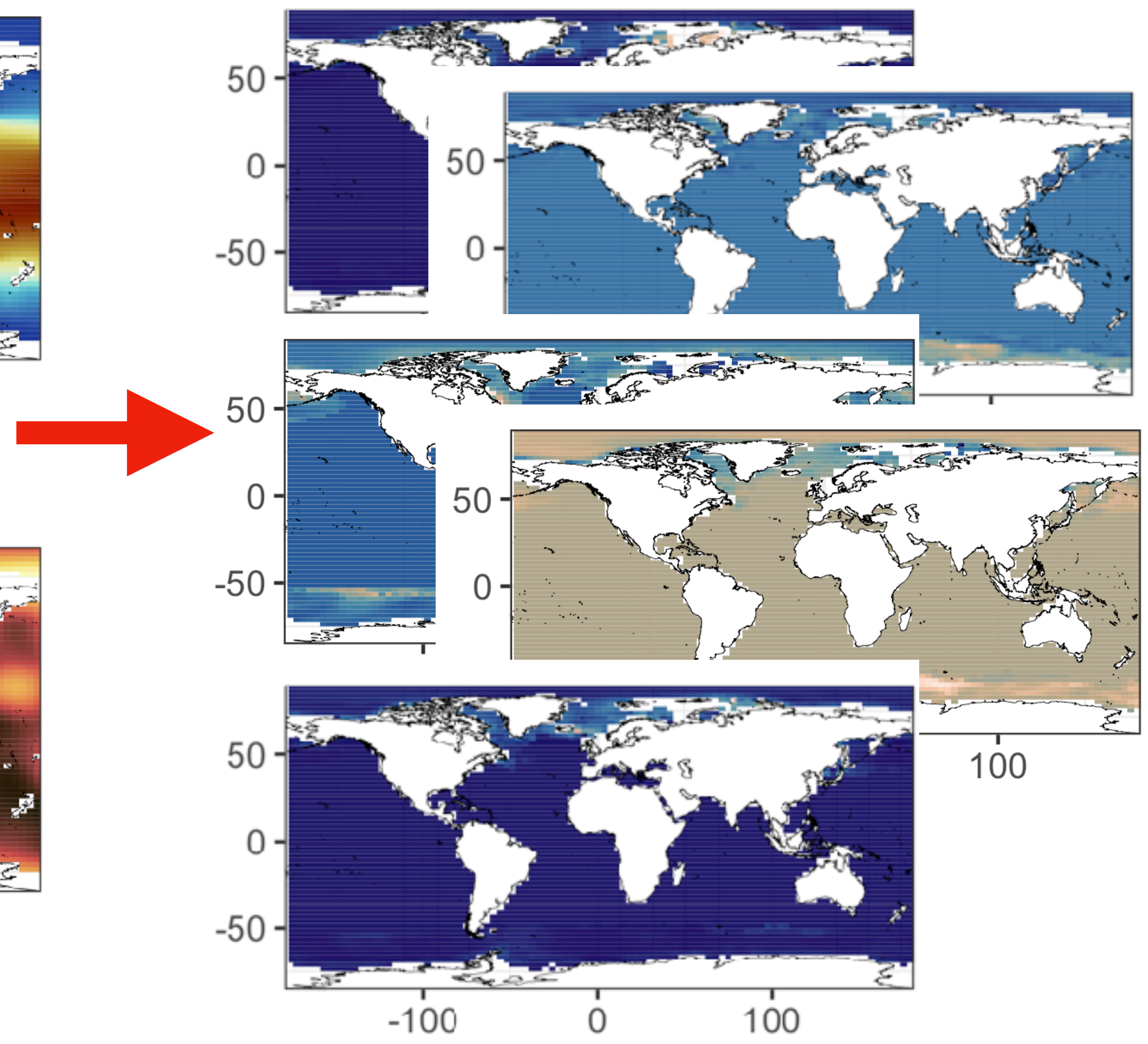


History Matching SIC

Generate
$\tilde{T}_X \sim \mathbb{E}_{X,Z}[T_X] + (\text{var}_{X,Z}[T_X])^{1/2}\epsilon$

Generate
$\tilde{\beta} \sim \mathbb{E}_{(X,Y)}[M(\beta)] + (\text{var}_{(X,Y)}[\beta])^{1/2}\epsilon$

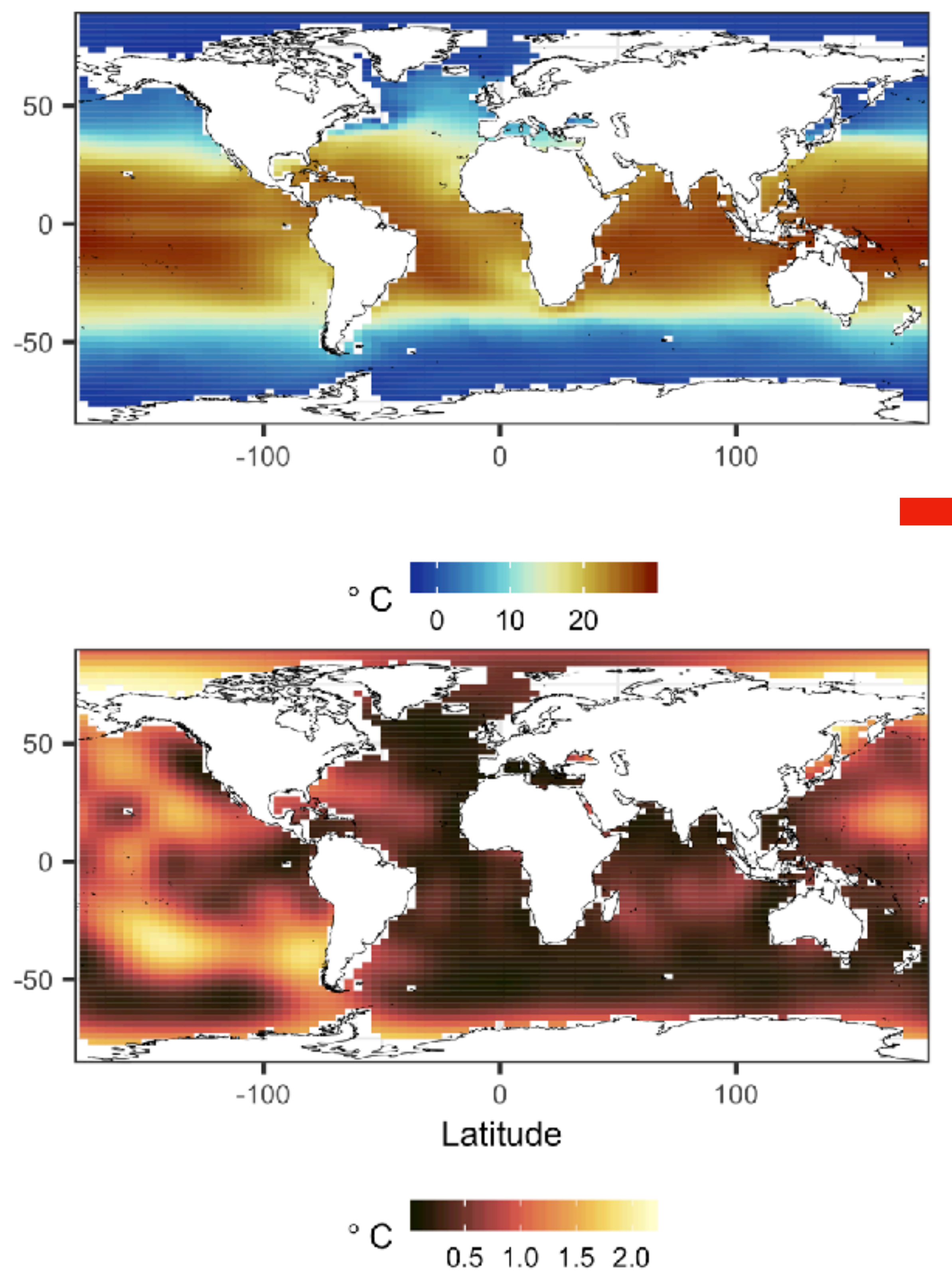




History Matching SIC

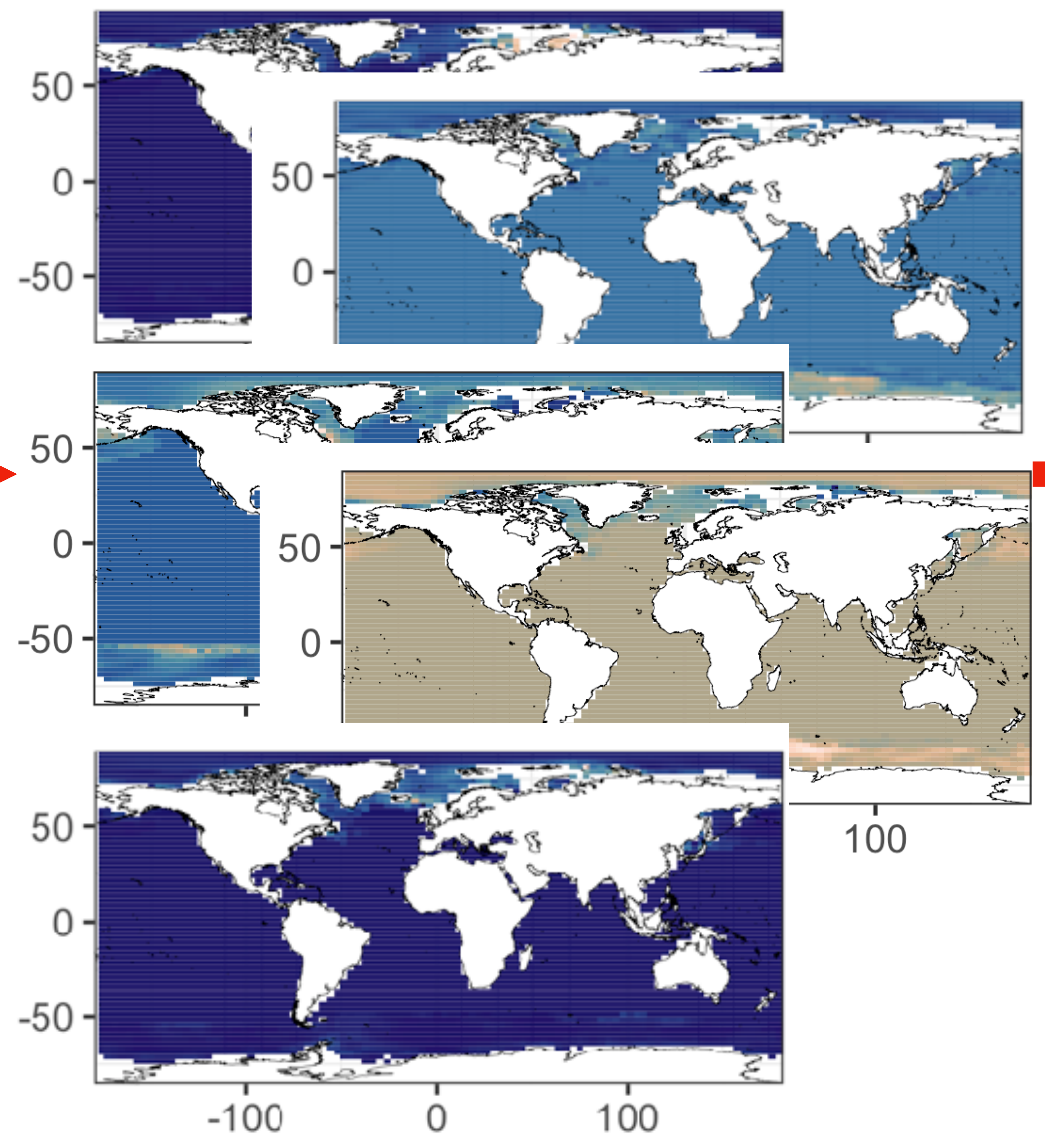
Generate

$$\tilde{T}_X \sim \mathbb{E}_{X,Z}[T_X] + (\text{var}_{X,Z}[T_X])^{1/2}\epsilon$$



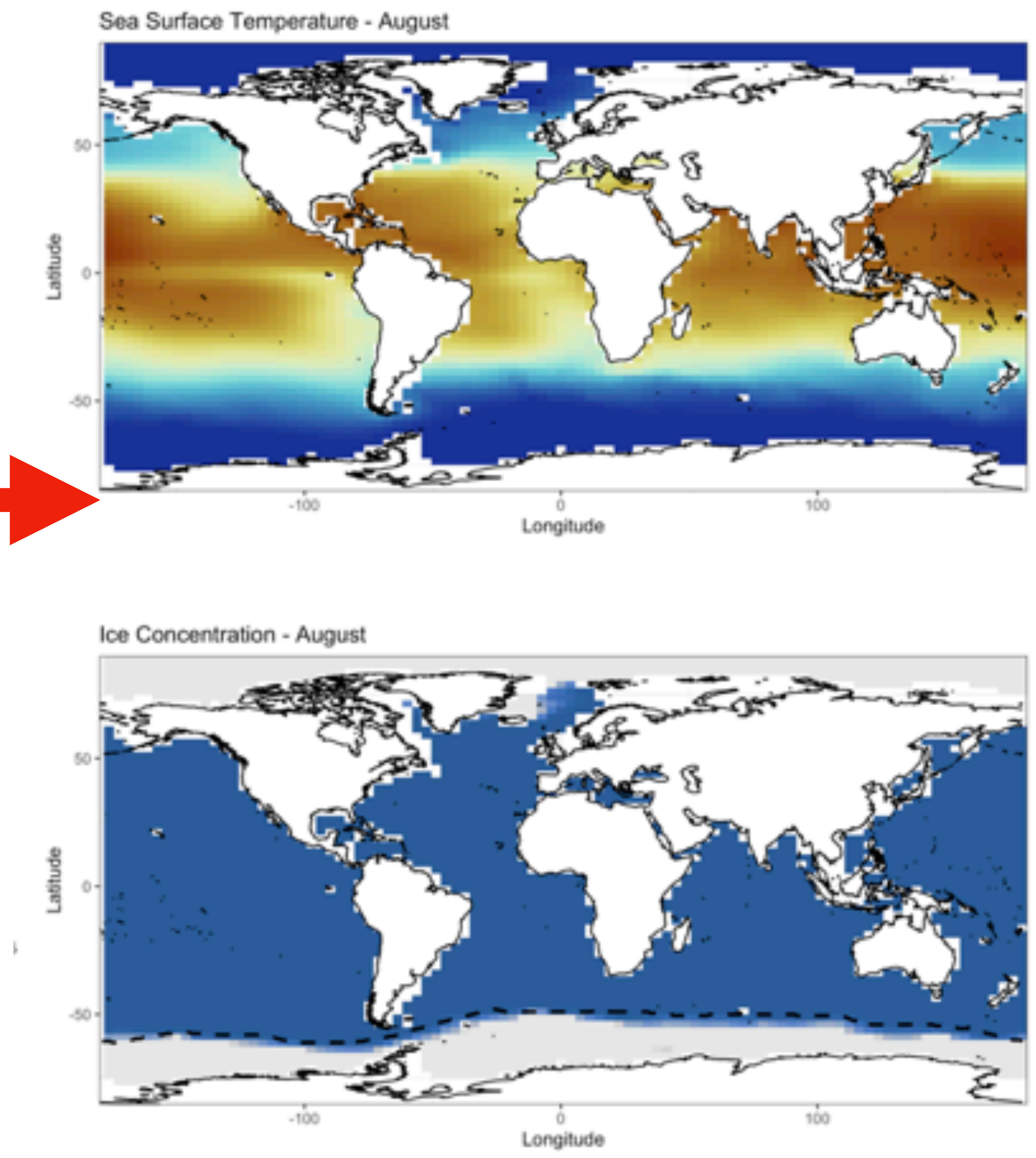
Generate

$$\tilde{\beta} \sim \mathbb{E}_{(X,Y)}[M(\beta)] + (\text{var}_{(X,Y)}[\beta])^{1/2}\epsilon$$



Generate

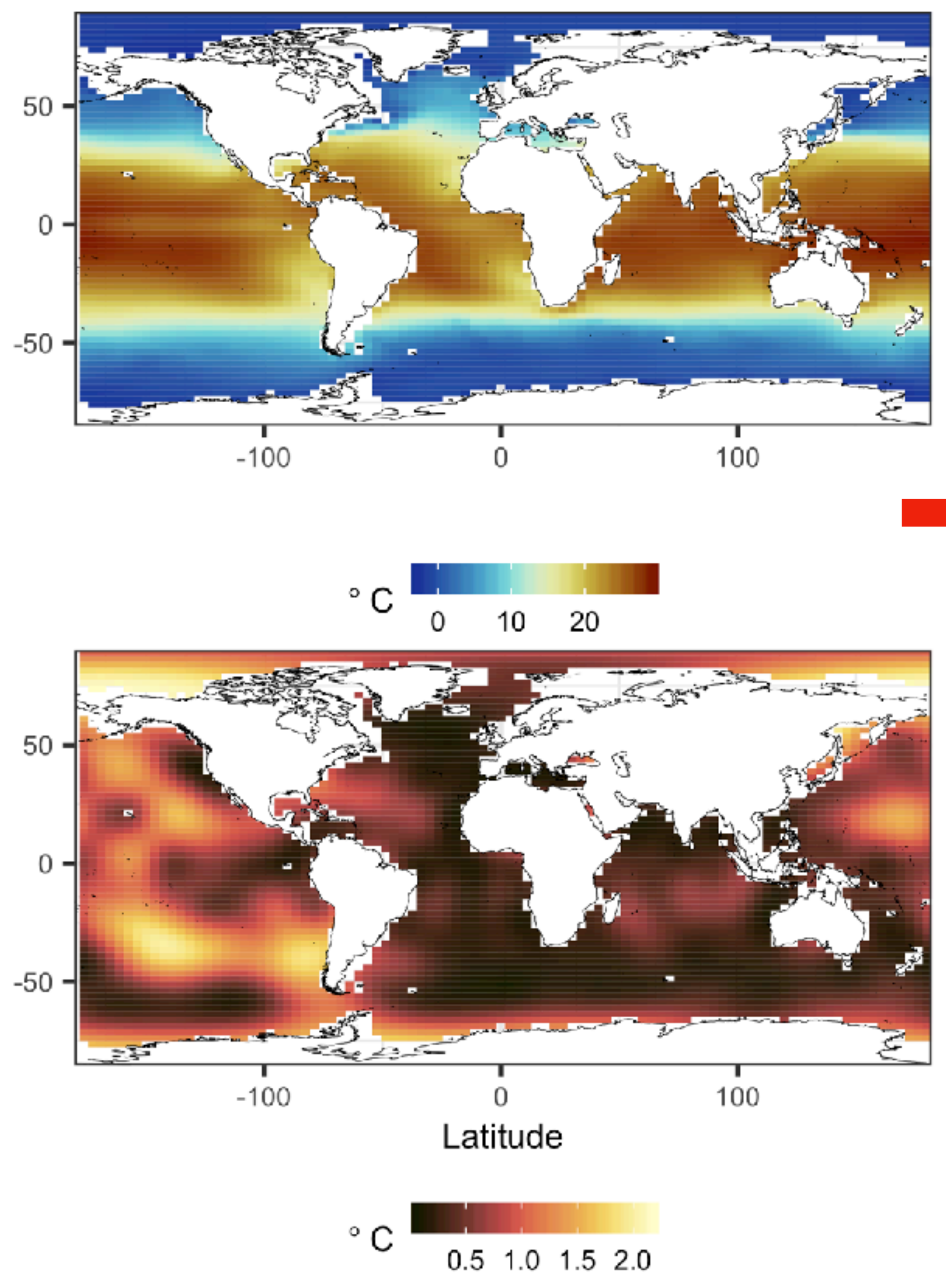
$$\tilde{T}_Y \sim \Phi_{\tilde{T}_X}\tilde{\beta} + (\text{var}[R(T_Y)])^{1/2}\epsilon$$



History Matching SIC

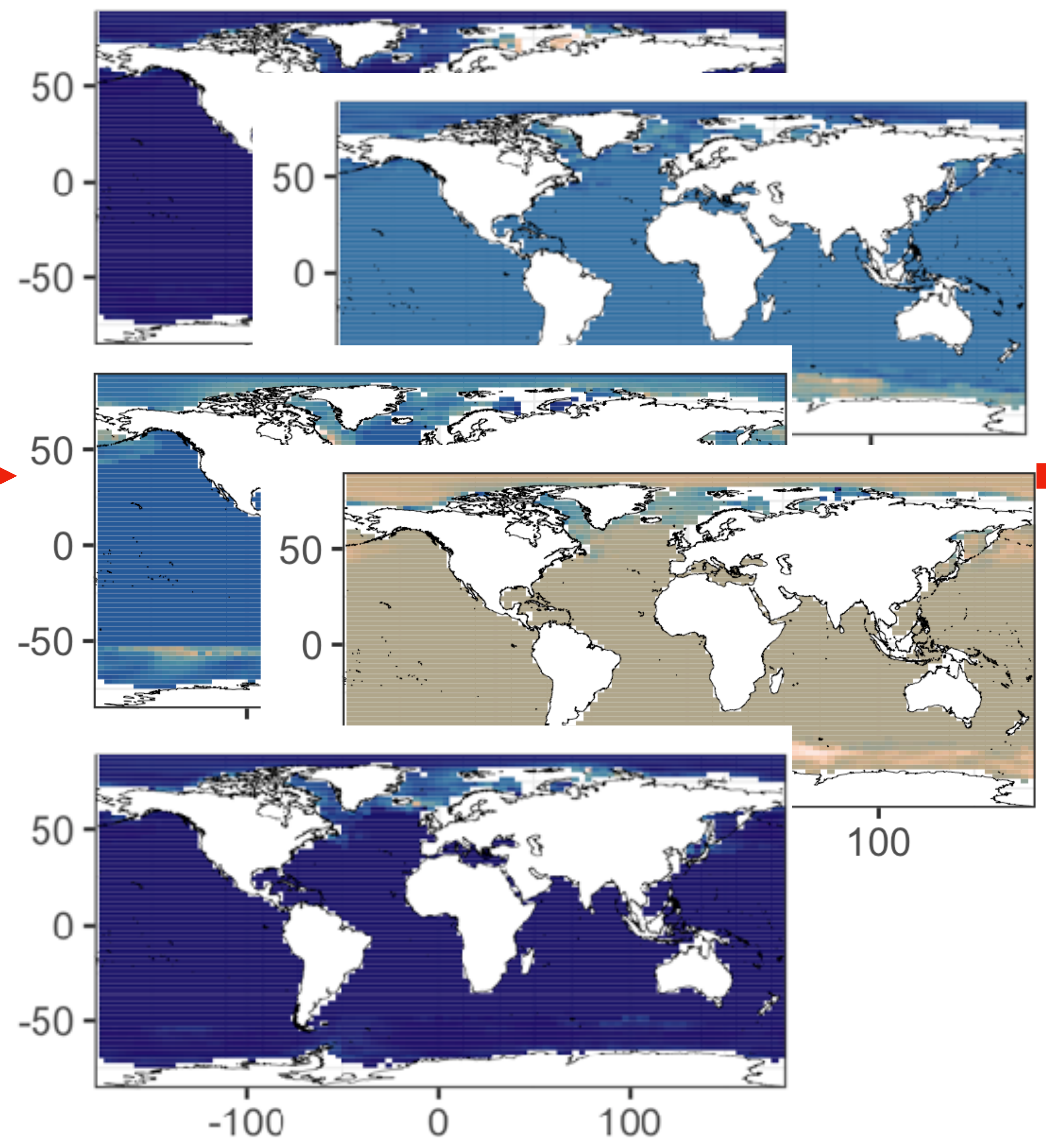
Generate

$\tilde{T}_X \sim \mathbb{E}_{X,Z}[T_X] + (\text{var}_{X,Z}[T_X])^{1/2}\epsilon$



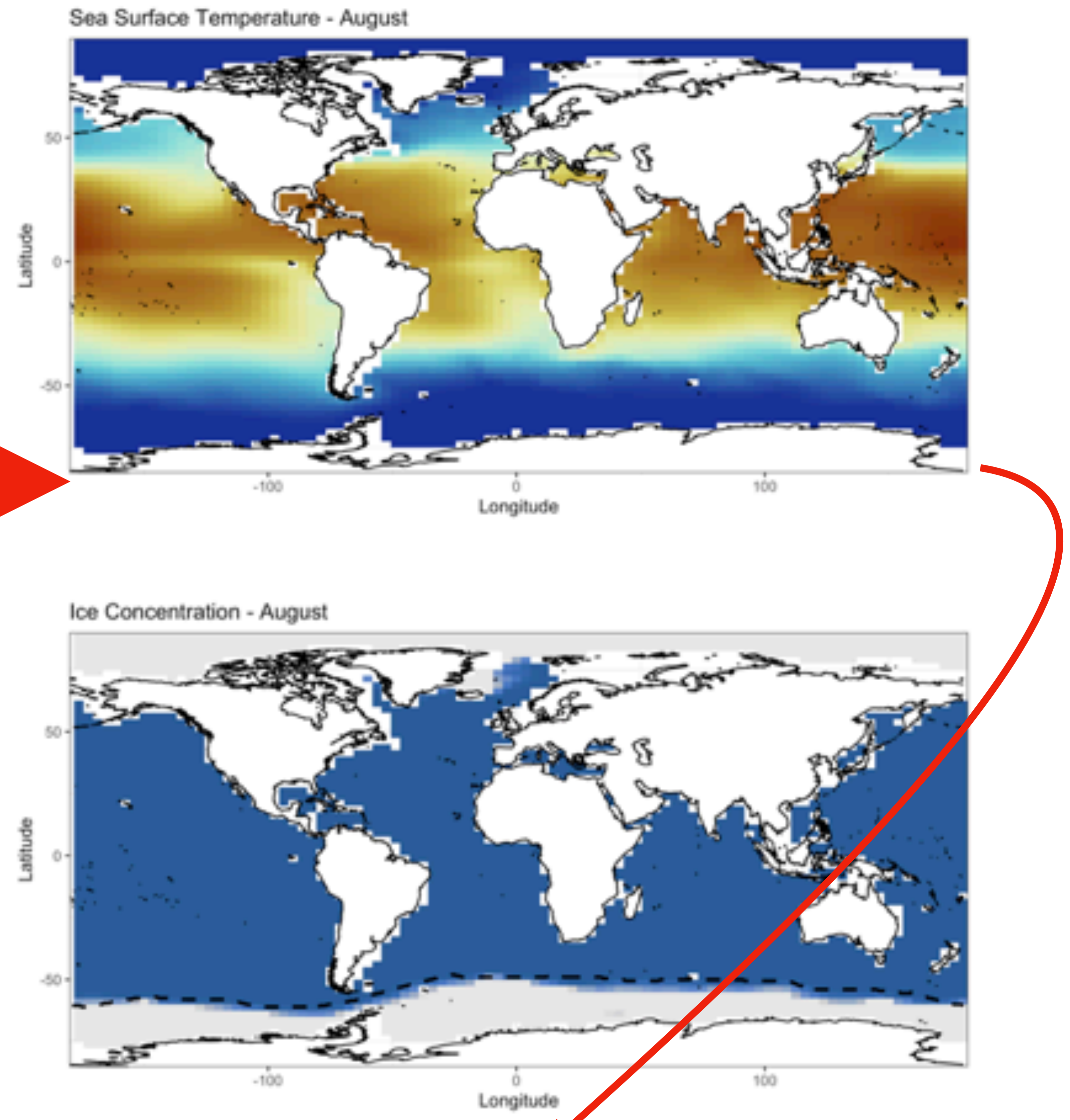
Generate

$\tilde{\beta} \sim \mathbb{E}_{(X,Y)}[M(\beta)] + (\text{var}_{(X,Y)}[\beta])^{1/2}\epsilon$



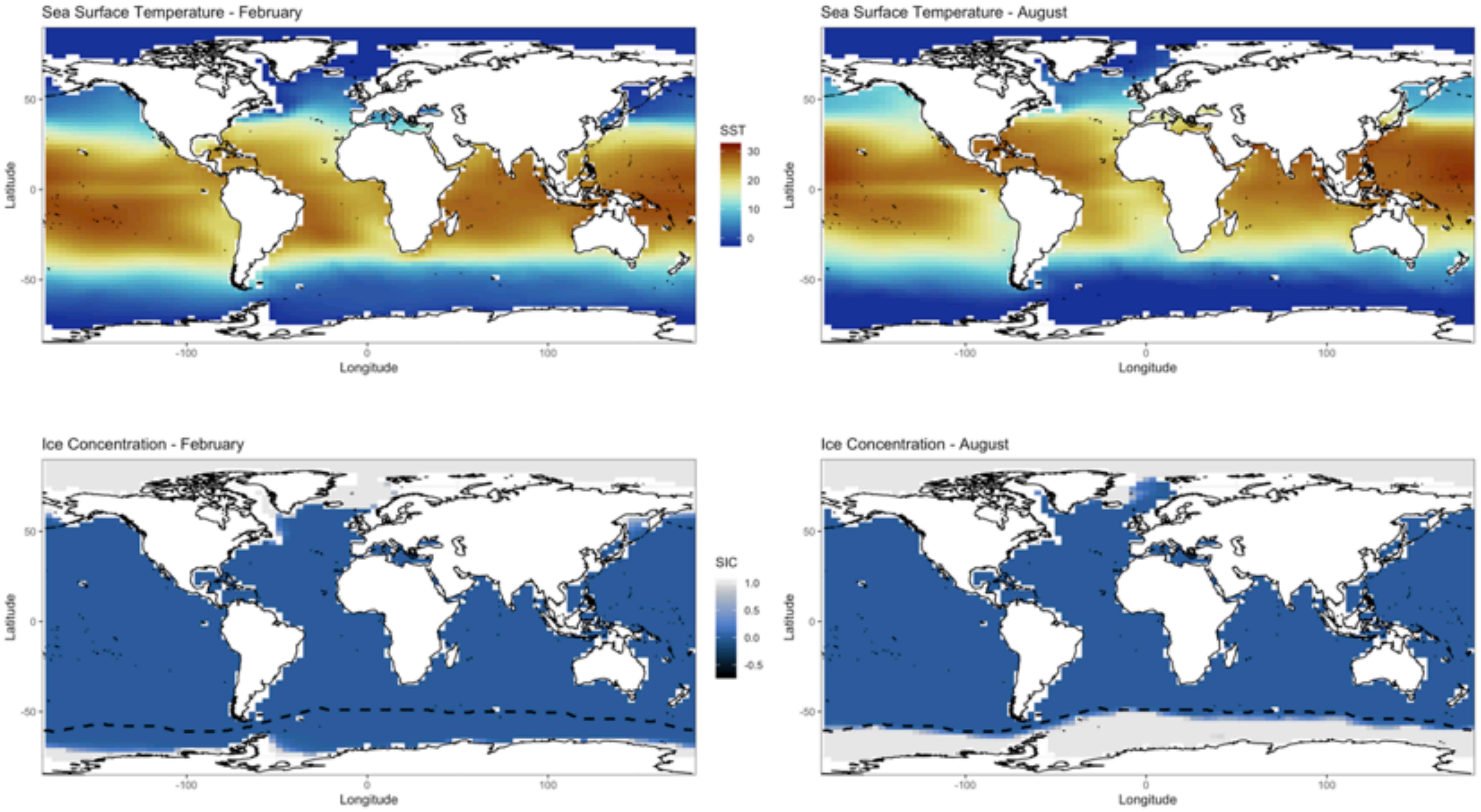
Generate

$\tilde{T}_Y \sim \Phi_{\tilde{T}_X}\tilde{\beta} + (\text{var}[R(T_Y)])^{1/2}\epsilon$



Check for membership in NROY space with sea-ice extent data

Joint reconstructions of SST and SIC



Thank you

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Some References

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Updating $M(\beta)$

As we have specified conditional (on X_i) exchangeability for the Y_i we may not immediately utilise Bayes linear sufficiency arguments for exchangeable data.

We may make sequential partial updates to our beliefs of $M(\beta)$, but we may also calculate this jointly:

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_m \\ \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \left[\begin{array}{ccc|c} \Phi_1 & \cdots & 0_{pn_1 \times k} & \\ \vdots & \ddots & \vdots & 0_{pm\bar{n} \times k} \\ 0_{pn_m \times k} & \cdots & \Phi_m & \\ \hline & 0_{km \times km} & & \mathbf{J}_{m \times 1} \otimes \mathbf{I}_k \end{array}\right] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \mathcal{M}(\beta) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_1(Y_1; X_1) \\ \vdots \\ \mathcal{R}_m(Y_m; X_m) \\ \mathcal{R}_1(\beta) \\ \vdots \\ \mathcal{R}_m(\beta) \end{bmatrix}$$

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Updating $M(\beta)$

Noting that $0 = M(\beta) - \beta_i + R_i(\beta)$ with some manipulation (as in Hodges (1998)) we may restate this to

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_m \\ 0_{km\times 1} \end{bmatrix} = \left[\begin{array}{ccc|c} \Phi_1 & \cdots & 0_{pn_1\times k} & \\ \vdots & \ddots & \vdots & 0_{pm\bar{n}\times k} \\ 0_{pn_m\times k} & \cdots & \Phi_m & \\ \hline & -\mathbf{I}_{km} & & \mathbf{J}_{m\times 1}\otimes \mathbf{I}_k \end{array}\right] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \mathcal{M}(\beta) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_1(Y_1;X_1) \\ \vdots \\ \mathcal{R}_m(Y_m;X_m) \\ \mathcal{R}_1(\beta) \\ \vdots \\ \mathcal{R}_m(\beta) \end{bmatrix}$$